Physics 704/804 Electromagnetic Theory II

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Scattering and Diffraction

$$\vec{E}_{inc} = \varepsilon_0 E_0 e^{ik\hat{n}_0\cdot\vec{x}}$$

$$\vec{H}_{inc} = \hat{n}_0 \times \vec{E}_{inc} / Z_0$$

$$\vec{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left[\left(\hat{n}\times\vec{p}\right)\times\hat{n} - \hat{n}\times\vec{m}/c\right] \quad \vec{H}_{sc} = \hat{n}\times\vec{E}_{sc} / Z_0$$

$$\frac{d\sigma}{d\Omega}\left(\hat{n},\hat{\varepsilon};\hat{n}_0,\hat{\varepsilon}_0\right) = \frac{r^2 \frac{1}{2Z_0}\left|\hat{\varepsilon}^* \cdot \vec{E}_{sc}\right|^2}{\frac{1}{2Z_0}\left|\hat{\varepsilon}_0^* \cdot \vec{E}_{inc}\right|^2}$$

Rayleigh's Law

$$\frac{d\sigma}{d\Omega}\left(\hat{n},\hat{\varepsilon};\hat{n}_0,\hat{\varepsilon}_0\right) = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2}\left|\hat{\varepsilon}^* \cdot \vec{p} - \left(\hat{n}\times\hat{\varepsilon}^*\right)\cdot\vec{m}/c\right|^2$$

Perfectly Conducting Sphere

$$\vec{p} = 4\pi\varepsilon_0 a^3 \vec{E}_{inc}; \qquad \vec{m} = -2\pi a^3 \vec{H}_{inc}$$

$$\frac{d\sigma}{d\Omega}\left(\hat{n},\hat{\varepsilon};\hat{n}_0,\hat{\varepsilon}_0\right) = k^4 a^6 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}\left(\hat{n}\times\hat{\varepsilon}^*\right)\cdot\left(\hat{n}\times\hat{\varepsilon}_0\right)\right|^2$$

$$\frac{d\sigma_{par}}{d\Omega} = \frac{k^4 a^6}{2}\left|\cos^2\theta - \frac{1}{2}\right|^2$$

$$\frac{d\sigma_{perp}}{d\Omega} = \frac{k^4 a^6}{2}\left|1 - \frac{1}{2}\cos^2\theta\right|^2$$

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta\right]$$

Expansion Vector Plane Wave

$$\frac{e^{ikr'}}{4\pi r'} e^{-ik\hat{n}\cdot\vec{x}} = ik \frac{e^{ikr'}}{4\pi r'} \sum_{l,m} (-i)^{l+1} j_l(kr) Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi)$$

$$\therefore e^{i\vec{k}\cdot\vec{x}} = 4\pi \sum_l (i)^l j_l(kr) \sum_{m=-l}^{l} Y_{lm}^*(\theta,\phi) Y_{lm}(\theta',\phi')$$

$$= \sum_{l=0}^{\infty} (i)^l (2l+1) j_l(kr) P_l(\cos\gamma)$$

$$= \sum_{l=0}^{\infty} (i)^l \sqrt{4\pi(2l+1)} j_l(kr) Y_{l0}(\gamma)$$

Circularly Polarized Plane Wave

$$\vec{E}\left(\vec{x}\right)=\left(\vec{\varepsilon}_1 \pm i\vec{\varepsilon}_1\right)e^{ikz}$$

$$c\vec{B}\left(\vec{x}\right)=\hat{z}\times\vec{E}=\mp i\vec{E}$$

Expansion as in Chapter 9

$$\vec{E}=\sum_{lm}\left[\frac{i}{k}b_\pm\left(l,m\right)\vec{\nabla}\times j_l\left(kr\right)\vec{X}_{lm}+a_\pm\left(l,m\right)j_l\left(kr\right)\vec{X}_{lm}\right]$$

$$c\vec{B}=\sum_{lm}\left[b_\pm\left(l,m\right)j_l\left(kr\right)\vec{X}_{lm}-\frac{i}{k}a_\pm\left(l,m\right)\vec{\nabla}\times j_l\left(kr\right)\vec{X}_{lm}\right]$$

$$\int\left[f_{l'}\left(r\right)\vec{X}_{l'm'}\right]^*\cdot\left[g_l\left(r\right)\vec{X}_{lm}\right]d\Omega=f_l^*g_l\delta_{ll'}\delta_{mm'}$$

$$\int\left[f_{l'}\left(r\right)\vec{X}_{l'm'}\right]^*\cdot\left(\vec{\nabla}\times g_l\left(r\right)\vec{X}_{lm}\right)d\Omega=0$$

Circularly Polarized Plane Wave

$$\frac{1}{k^2}\int\left[\vec{\nabla}\times f_{l'}(r)\vec{X}_{l'm'}\right]^*\cdot\left(\vec{\nabla}\times g_l(r)\vec{X}_{lm}\right)d\Omega$$

$$=\delta_{l'l}\delta_{m'm}\left\{f_l^* g_l+\frac{1}{k^2r^2}\frac{\partial}{\partial r}\left[rf_l^*\frac{\partial}{\partial r}(rg_l)\right]\right\}$$

$$a_\pm(l,m)\,j_l(kr)=\int\vec{X}_{lm}^*\cdot\vec{E}(\vec{x})\,d\Omega$$

$$a_\pm(l,m)\,j_l(kr)=\int\frac{\left(L_\mp Y_{lm}\right)^*}{\sqrt{l(l+1)}}e^{ikz}d\Omega$$

$$=\frac{\sqrt{(l\pm m)(l\mp m+1)}}{\sqrt{l(l+1)}}\int Y_{l,m\mp 1}^* e^{ikz}d\Omega$$

$$a_\pm(l,m)=i^l\sqrt{4\pi(2l+1)}\,\delta_{m,\pm 1}$$

Scattering by a Sphere

$$b_{\pm}(l,m)\, j_l(kr) = c\int \vec{X}^{*}_{lm} \cdot \vec{B}(\vec{x})\, d\Omega \rightarrow b_{\pm}(l,m) = \mp i a_{\pm}(l,m)$$

$$\vec{E} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left[j_l(kr)\, \vec{X}_{l,\pm 1} \pm \frac{1}{k} \vec{\nabla} \times j_l(kr)\, \vec{X}_{l,\pm 1} \right]$$

$$c\vec{B} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left[-\frac{i}{k} \vec{\nabla} \times j_l(kr)\, \vec{X}_{l,\pm 1} \mp i j_l(kr)\, \vec{X}_{l,\pm 1} \right]$$

Scattered Wave

$$\vec{E} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left[\alpha_{l\pm} h_l^1(kr)\, \vec{X}_{l,\pm 1} \pm \beta_{l\pm} \frac{1}{k} \vec{\nabla} \times h_l^1(kr)\, \vec{X}_{l,\pm 1} \right]$$

$$c\vec{B} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)} \left[-\frac{i\alpha_{l\pm}}{k} \vec{\nabla} \times h_l^1(kr)\, \vec{X}_{l,\pm 1} \mp i \beta_{l\pm} h_l^1(kr)\, \vec{X}_{l,\pm 1} \right]$$

Scattering Cross Section

$$\sigma_{sc} = \frac{\pi}{2k^2}\sum_l (2l+1)\left[\left|\alpha_l\right|^2 + \left|\beta_l\right|^2\right]$$

Differential Cross Section

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{\pi}{2k^2}\left|\sum_l \sqrt{2l+1}\left[\alpha_{l\pm}\vec{X}_{l,\pm1} \pm i\beta_{l\pm}\hat{n}\times\vec{X}_{l,\pm1}\right]\right|^2$$

For Perfect Conducting Sphere in long-wavelength limit

$$\alpha_{1\pm} = -\frac{1}{2}\beta_{1\pm} \approx -\frac{2i}{3}(ka)^3$$

$$\frac{d\sigma_{sc}}{d\Omega} \approx \frac{2\pi}{3}a^2(ka)^4\left|\left[\vec{X}_{1,\pm1} \pm i2\hat{n}\times\vec{X}_{1,\pm1}\right]\right|^2$$

$$\left|\hat{n}\times\vec{X}_{1,\pm1}\right|^2 = \left|\vec{X}_{1,\pm1}\right|^2 = \frac{3}{16\pi}\left(1+\cos^2\theta\right) \quad \left[\pm i\left(\hat{n}\times\vec{X}_{1,\pm1}\right)^*\cdot\vec{X}_{1,\pm1}\right] = -\frac{3}{8\pi}\cos\theta$$

$$\frac{d\sigma_{sc}}{d\Omega} \approx a^2(ka)^4\left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta\right]$$

Special Relativity

Two Postulates

I Relativity Principle: Laws of nature and results of all experiments the same in all inertial reference frames.

II The velocity of light measured by an observer is independent of the velocity of the source

Invariance of the interval

$$c^2 t'^2 - x'^2 - y'^2 - z'^2 = c^2 t^2 - x^2 - y^2 - z^2$$

$$x^0 = ct$$

$$x^1 = x$$

$$x^2 = y$$

$$x^3 = z$$

Lorentz Transformations

$$x'^0 = \gamma\left(x^0 - \beta x^1\right)$$

$$x'^1 = \gamma\left(x^1 - \beta x^0\right)$$

$$x'^2 = x^2$$

$$x'^2 = x^2$$

4-Vectors and Proper Time

Any set of four quantities that transform as the coordinate displacements is called a (contravariant) 4-vector

$$x'^0 = \gamma\left(x^0 - \beta x^1\right)$$

$$x'^1 = \gamma\left(x^1 - \beta x^0\right)$$

$$x'^2 = x^2$$

$$x'^3 = x^3$$

proper (invariant!) time

$$d\tau = \int \sqrt{1 - \vec{v}\cdot\vec{v}/c^2}\,dt$$

Invariance of wave phase

$$\omega' t' - \vec{k}' \cdot \vec{x}' = \omega t - \vec{k} \cdot \vec{x}$$

$\left(\omega, \vec{k}c\right)$ for photons is a 4-vector

$$\omega' = \gamma\left(\omega - \beta k_z c\right)$$

$$k_z = \gamma\left(k_z - \beta\omega / c\right)$$

$$k'_x = k_x$$

$$k'_y = k_y$$

relativistic Doppler Shift

$$\omega' = \gamma\omega\left(1 - \beta\cos\theta\right)$$

$$\tan\theta' = \frac{\sin\theta'}{\cos\theta'} = \frac{\sin\theta}{\gamma\left(\cos\theta - \beta\right)}$$

Other 4-vectors

velocity 4-vector

$$U = \left(\frac{dt}{d\tau} c, \frac{d\vec{x}}{d\tau} \right) = c\left(\gamma, \gamma\vec{\beta}\right)$$

$$U \cdot U = c^2$$

Energy-momentum 4-vector

$$p = mU$$

$$p \cdot p = m^2 c^2 = \frac{E^2}{c^2} - m^2 \vec{v} \cdot \vec{v}$$

$$E^2 = \left|\vec{p}\right|^2 c^2 + m^2 c^4$$