Physics 704/804 Electromagnetic Theory II

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Multipole Sources

$$\vec{\nabla} \cdot \frac{\vec{B}}{\mu} = 0 \qquad \vec{\nabla} \times \vec{E} - ikZ_0 \frac{\vec{B}}{\mu} = 0$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \frac{\vec{B}}{\mu_0} + ik\vec{E} / Z_0 = \vec{J} + \vec{\nabla} \times \vec{M}$$

$$i\omega\rho = \vec{\nabla} \cdot \vec{J} \qquad \vec{H}' = \frac{\vec{B}}{\mu_0} \qquad \vec{E}' = \vec{E} + \frac{i}{\omega\varepsilon_0}\vec{J}$$

$$\vec{\nabla} \cdot \vec{H}' = 0 \qquad \vec{\nabla} \times \vec{E} - ikZ_0 \frac{\vec{B}}{\mu} = \frac{i}{\omega\varepsilon_0}\vec{\nabla} \times \vec{J}$$

equations are now like before

Inhomogeneous Helmholtz Equation

$$\left(\nabla^2 + k^2\right)\vec{r}\cdot\vec{H}' = -i\vec{L}\cdot\left(\vec{J} + \vec{\nabla}\times\vec{M}\right)$$

$$\left(\nabla^2 + k^2\right)\vec{r}\cdot\vec{E}' = Z_0 k\vec{L}\cdot\left(\vec{M} + \frac{1}{k^2}\vec{\nabla}\times\vec{J}\right)$$

$$\vec{r}\cdot\vec{H}' = \frac{i}{4\pi}\int\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}\vec{L}'\cdot\left[\vec{J} + \vec{\nabla}'\times\vec{M}\right]d^3\vec{x}'$$

$$\vec{r}\cdot\vec{E}' = -\frac{Z_0 k}{4\pi}\int\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}\vec{L}'\cdot\left[\vec{M} + \frac{1}{k^2}\vec{\nabla}'\times\vec{J}\right]d^3\vec{x}'$$

$$\frac{1}{4\pi}\int d\Omega\, Y_{lm}^*(\theta,\phi)\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|} = ikh_l^1(kr)\, j_l(kr')Y_{lm}^*(\theta,\phi)$$

$$a_E(l,m) = \frac{ik^3}{\sqrt{l(l+1)}} \int j_l(kr) Y_{lm}^* \vec{L} \cdot \left(\vec{M} + \frac{1}{k^2} \vec{\nabla} \times \vec{J} \right) d^3\vec{x}$$

$$a_M(l,m) = -\frac{k^2}{\sqrt{l(l+1)}} \int j_l(kr) Y_{lm}^* \vec{L} \cdot \left(\vec{J} + \vec{\nabla} \times \vec{M} \right) d^3\vec{x}$$

$$a_E(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^* \left\{ \begin{array}{c} c\rho \frac{\partial}{\partial r}\left[r j_l(kr) \right] + ik\left(\vec{r} \cdot \vec{J} \right) j_l(kr) \\ - ik\vec{\nabla} \cdot \left(\vec{r} \times \vec{M} \right) j_l(kr) \end{array} \right\} d^3\vec{x}$$

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^* \left\{ \begin{array}{c} \vec{\nabla} \cdot \left(\vec{r} \times \vec{J} \right) j_l(kr) \\ + \left(\vec{r} \cdot \vec{M} \right) \frac{\partial}{\partial r}\left[r j_l(kr) \right] \\ - k^2 \left(\vec{r} \cdot \vec{M} \right) j_l(kr) \end{array} \right\} d^3\vec{x}$$

Small Source Moments

for small sources

$$a_E(l,m) = \frac{ck^{l+2}}{i(2l+1)!!}\sqrt{\frac{l+1}{l}}\left(Q_{lm} + Q'_{lm}\right)$$

$$a_M(l,m) = \frac{ick^{l+2}}{(2l+1)!!}\sqrt{\frac{l+1}{l}}\left(M_{lm} + M'_{lm}\right)$$

$$Q_{lm} = \int r^l Y^*_{lm}\rho\, d^3x \qquad Q'_{lm} = \frac{-ik}{(l+1)c}\int r^l Y^*_{lm}\vec{\nabla}\cdot\left(\vec{r}\times\vec{M}\right)d^3x$$

$$M'_{lm} = -\int r^l Y^*_{lm}\vec{\nabla}\cdot\vec{M}\, d^3x \qquad M_{lm} = -\frac{1}{(l+1)}\int r^l Y^*_{lm}\vec{\nabla}\cdot\left(\vec{r}\times\vec{J}\right)d^3x$$

Scattering and Diffraction

$$\vec{E}_{inc} = \varepsilon_0 E_0 e^{ik\hat{n}_0 \cdot \vec{x}}$$

$$\vec{H}_{inc} = \hat{n}_0 \times \vec{E}_{inc} / Z_0$$

$$\vec{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r} \left[\left(\hat{n} \times \vec{p} \right) \times \hat{n} - \hat{n} \times \vec{m} / c \right] \quad \vec{H}_{sc} = \hat{n} \times \vec{E}_{sc} / Z_0$$

$$\frac{d\sigma}{d\Omega}\left(\hat{n}, \hat{\varepsilon}; \hat{n}_0, \hat{\varepsilon}_0\right) = \frac{r^2 \frac{1}{2Z_0} \left| \hat{\varepsilon}^* \cdot \vec{E}_{sc} \right|^2}{\frac{1}{2Z_0} \left| \hat{\varepsilon}_0^* \cdot \vec{E}_{inc} \right|^2}$$

Rayleigh's Law

$$\frac{d\sigma}{d\Omega}\left(\hat{n}, \hat{\varepsilon}; \hat{n}_0, \hat{\varepsilon}_0\right) = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left| \hat{\varepsilon}^* \cdot \vec{p} - \left(\hat{n} \times \hat{\varepsilon}^* \right) \cdot \vec{m} / c \right|^2$$

Small Sphere Scattering

dielectric sphere

$$\vec{p} = 4\pi\varepsilon_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right) a^3 \vec{E}_{inc}$$

$$\frac{d\sigma}{d\Omega}\left(\hat{n}, \hat{\varepsilon}; \hat{n}_0, \hat{\varepsilon}_0\right) = \frac{k^4 a^6}{\left(4\pi\varepsilon_0 E_0\right)^2} \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 \right|^2$$

$$\frac{d\sigma_{par}}{d\Omega}\left(\hat{n}, \hat{\varepsilon}; \hat{n}_0, \hat{\varepsilon}_0\right) = \frac{k^4 a^6}{2} \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 \cos^2\theta$$

$$\frac{d\sigma_{perp}}{d\Omega}\left(\hat{n}, \hat{\varepsilon}; \hat{n}_0, \hat{\varepsilon}_0\right) = \frac{k^4 a^6}{2} \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2$$

Total Cross Section

net polarization

$$\Pi(\theta) = \frac{\frac{d\sigma_{perp}}{d\Omega} - \frac{d\sigma_{par}}{d\Omega}}{\frac{d\sigma_{perp}}{d\Omega} + \frac{d\sigma_{par}}{d\Omega}} = \frac{\sin^2\theta}{1+\cos^2\theta}$$

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \frac{1}{2}\left(1+\cos^2\theta\right)$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} k^4 a^6 \left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2$$