Physics 704/804 Electromagnetic Theory II

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Applied to Spherical Harmonics

$$L_+ Y_{lm} = \sqrt{(l-m)(l+m+1)}Y_{lm}$$

$$L_- Y_{lm} = \sqrt{(l+m)(l-m+1)}Y_{lm}$$

$$L_z Y_{lm} = m Y_{lm}$$

$$L^2 \vec{L} = \vec{L} L^2$$

$$\vec{L} \times \vec{L} = i\vec{L}$$

$$L_j \nabla^2 = \nabla^2 L_j$$

Vector Spherical Harmonics

$$\vec{X}_{lm} = \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{lm}(\theta, \phi)$$

$$\int \vec{X}^*_{l'm'} \cdot \vec{X}_{lm} \, d\Omega = \delta_{ll'} \delta_{mm'}$$

Did exercises in Chapter 12.11 of Arfken (3rd Edition) and obtained a lot of clarity

Show $\vec{X}_{lm} = \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{lm} =$

$$\hat{\theta}\left\{\frac{-m}{\sqrt{l(l+1)}\sin\theta} Y_{lm}\right\} + \hat{\phi}\left\{\frac{-i}{\sqrt{l(l+1)}} \frac{\partial Y_{lm}}{\partial \theta}\right\}$$

Verify the orthogonality relation

$$\int \vec{X}^{*}_{l'm'} \cdot \vec{X}_{lm} \, d\Omega = \delta_{ll'}\delta_{mm'}$$

Verify the sum rule

$$\sum_{m=-l}^{l} \left| \vec{X}_{lm}(\theta,\phi) \right|^2 = \frac{2l+1}{4\pi}$$

Show power for pure multipole formula from last time:

$$\frac{dP(l,m)}{d\Omega} = \frac{Z_0}{2k^2} \left| a(l,m) \right|^2 \left| \vec{X}_{lm} \right|^2$$

$$= \frac{Z_0}{2k^2 l(l+1)} \left| a(l,m) \right|^2 \left\{ \begin{array}{c} \frac{1}{2}(l-m)(l+m+1)\left|Y_{l,m+1}\right|^2 \\ +\frac{1}{2}(l+m)(l-m+1)\left|Y_{l,m-1}\right|^2 + m^2\left|Y_{l,m}\right|^2 \end{array} \right\}$$

Magnetic Multipole Field (TE)

$$\left(\vec{r}\cdot\vec{H}^{M}_{lm}\right)=\frac{l\left(l+1\right)}{k}g_l\left(kr\right)Y_{lm}\left(\theta,\phi\right)$$

$$\left(\vec{r}\cdot\vec{E}^{M}_{lm}\right)=0$$

$$Z_0 k\vec{r}\cdot\vec{H}=\frac{1}{i}r\cdot\left(\vec{\nabla}\times\vec{E}\right)=\frac{1}{i}\left(\vec{r}\times\vec{\nabla}\right)\cdot E=L\cdot\vec{E}$$

$$\vec{L}\cdot\vec{E}^{M}_{lm}\left(r,\theta,\phi\right)=l\left(l+1\right)Z_0 g_l\left(r\right)Y_{lm}\left(\theta,\phi\right)$$

$$\vec{E}^{M}_{lm}=Z_0 g_l\left(r\right)\vec{L}Y_{lm}\left(\theta,\phi\right)$$

$$\vec{H}^{M}_{lm}=-\frac{i}{kZ_0}\vec{\nabla}\times\vec{E}^{M}_{lm}$$

General Solution

$$\vec{H} = \sum_{lm} \left[a_E(l,m) f_l(kr) \vec{X}_{lm} - \frac{i}{k} a_M(l,m) \vec{\nabla} \times g_l(kr) \vec{X}_{lm} \right]$$

$$\vec{E} = \sum_{lm} \left[\frac{i}{k} a_E(l,m) \vec{\nabla} \times f_l(kr) \vec{X}_{lm} + a_M(l,m) g_l(kr) \vec{X}_{lm} \right]$$

$$a_M(l,m) g_l(kr) = \frac{k}{\sqrt{l(l+1)}} \int Y_{lm}^* \vec{r} \cdot \vec{H} d\Omega$$

$$Z_0 a_M(l,m) g_l(kr) = -\frac{k}{\sqrt{l(l+1)}} \int Y_{lm}^* \vec{r} \cdot \vec{E} d\Omega$$

$\therefore$ knowledge at 2 different radii sufficient for complete knowledge of solution

Multipole Sources

$$\vec{\nabla}\cdot\frac{\vec{B}}{\mu}=0 \qquad \vec{\nabla}\times\vec{E}-ikZ_0\frac{\vec{B}}{\mu}=0$$

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\frac{\vec{B}}{\mu_0}+ik\vec{E}/Z_0=\vec{J}+\vec{\nabla}\times\vec{M}$$

$$i\omega\rho=\vec{\nabla}\cdot\vec{J} \qquad \vec{H}'=\frac{\vec{B}}{\mu_0} \qquad \vec{E}'=\vec{E}+\frac{i}{\omega\varepsilon_0}\vec{J}$$

$$\vec{\nabla}\cdot\vec{H}'=0 \qquad \vec{\nabla}\times\vec{E}-ikZ_0\frac{\vec{B}}{\mu}=\frac{i}{\omega\varepsilon_0}\vec{\nabla}\times\vec{J}$$

equations are now like before

Inhomogeneous Helmholtz Equation

$$\left(\nabla^2 + k^2\right)\vec{r}\cdot\vec{H}' = -i\vec{L}\cdot\left(\vec{J} + \vec{\nabla}\times\vec{M}\right)$$

$$\left(\nabla^2 + k^2\right)\vec{r}\cdot\vec{E}' = Z_0 k\vec{L}\cdot\left(\vec{M} + \frac{1}{k^2}\vec{\nabla}\times\vec{J}\right)$$

$$\vec{r}\cdot\vec{H}' = \frac{i}{4\pi}\int\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\vec{L}'\cdot\left[\vec{J} + \vec{\nabla}'\times\vec{M}\right]d^3\vec{x}'$$

$$\vec{r}\cdot\vec{E}' = -\frac{Z_0 k}{4\pi}\int\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\vec{L}'\cdot\left[\vec{M} + \frac{1}{k^2}\vec{\nabla}'\times\vec{J}\right]d^3\vec{x}'$$

$$\frac{1}{4\pi}\int d\Omega\, Y_{lm}^*(\theta,\phi)\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = ikh_l^1(kr)\, j_l(kr')\, Y_{lm}^*(\theta,\phi)$$