Physics 704/804 Electromagnetic Theory II

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

Multipole Expansion

$$\left(\nabla^2 + k^2\right)\vec{H} = 0 \qquad \vec{\nabla}\cdot\vec{H} = 0 \qquad \vec{E} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}$$

$$\left(\nabla^2 + k^2\right)\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{H} = -\frac{i}{kZ_0}\vec{\nabla}\times\vec{E}$$

$$\nabla^2\left(\vec{r}\cdot\vec{A}\right) = \vec{r}\cdot\left(\nabla^2\vec{A}\right) + 2\vec{\nabla}\cdot\vec{A}$$

$$\left(\nabla^2 + k^2\right)\left(\vec{r}\cdot\vec{H}\right) = 0 \qquad \left(\nabla^2 + k^2\right)\left(\vec{r}\cdot\vec{E}\right) = 0$$

Magnetic Multipole Field (TE)

$$\left(\vec{r}\cdot\vec{H}_{lm}^{M}\right)=\frac{l\left(l+1\right)}{k}g_{l}\left(kr\right)Y_{lm}\left(\theta,\phi\right)$$

$$\left(\vec{r}\cdot\vec{E}_{lm}^{M}\right)=0$$

$$Z_{0}k\vec{r}\cdot\vec{H}=L\cdot\vec{E}$$

$$\vec{L}\cdot\vec{E}_{lm}^{M}\left(r,\theta,\phi\right)=l\left(l+1\right)Z_{0}g_{l}\left(r\right)Y_{lm}\left(\theta,\phi\right)$$

$$\vec{E}_{lm}^{M}=Z_{0}g_{l}\left(r\right)\vec{L}Y_{lm}\left(\theta,\phi\right)$$

$$\vec{H}_{lm}^{M}=-\frac{i}{kZ_{0}}\vec{\nabla}\times\vec{E}_{lm}^{M}$$

Electric Multipole (TM)

$$\left(\vec{r}\cdot\vec{E}_{lm}^{E}\right) = -Z_0 \frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta,\phi)$$

$$\left(\vec{r}\cdot\vec{H}_{lm}^{E}\right) = 0$$

$$\vec{H}_{lm}^{E} = f_l(r)\vec{L}Y_{lm}(\theta,\phi)$$

$$\vec{E}_{lm}^{E} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H}_{lm}^{E}$$

normal mode expansions

$$\vec{X}_{lm} = \frac{1}{\sqrt{l(l+1)}}\vec{L}Y_{lm}(\theta,\phi)$$

$$\int \vec{X}_{l'm'}^{*}\cdot\vec{X}_{lm}\,d\Omega = \delta_{ll'}\delta_{mm'}$$

$$\int \vec{X}_{l'm'}^{*}\cdot\left(\vec{r}\times\vec{X}_{lm}\right)d\Omega = 0$$

General Solution

$$\vec{H} = \sum_{lm}\left[a_E(l,m) f_l(kr)\vec{X}_{lm} - \frac{i}{k}a_M(l,m)\vec{\nabla}\times g_l(kr)\vec{X}_{lm}\right]$$

$$\vec{E} = \sum_{lm}\left[\frac{i}{k}a_E(l,m)\vec{\nabla}\times f_l(kr)\vec{X}_{lm} + a_M(l,m) g_l(kr)\vec{X}_{lm}\right]$$

$$a_M(l,m) g_l(kr) = \frac{k}{\sqrt{l(l+1)}}\int Y_{lm}^{*}\vec{r}\cdot\vec{H}\,d\Omega$$

$$Z_0 a_E(l,m) g_l(kr) = -\frac{k}{\sqrt{l(l+1)}}\int Y_{lm}^{*}\vec{r}\cdot\vec{E}\,d\Omega$$

$\therefore$ knowledge at 2 different radii sufficient for complete knowledge of solution

Energy

$$\vec{H}_l = \sum_m a_E(l,m)\,\vec{X}_{lm} h_l^1(kr)\, e^{-i\omega t}$$

$$\vec{E}_l = \frac{i}{k} Z_0 \vec{\nabla} \times \vec{H}_l$$

$$u = \frac{\varepsilon_0}{4}\left(\vec{E}\cdot\vec{E}^* + Z_0^2 \vec{H}\cdot\vec{H}^*\right) \approx \frac{\mu_0}{2}\vec{H}\cdot\vec{H}^*$$

in radiation zone

$$dU = \frac{\mu_0 dr}{2k^2}\sum_{mm'} a_E^*(l,m)\, a_E(l,m')\int \vec{X}^*_{lm'}\vec{X}_{lm}\, d\Omega$$

$$\frac{dU}{dr} = \frac{\mu_0}{2k^2}\sum_{mm'}\left|a_E(l,m)\right|^2$$

incoherent sum over multipoles

Angular Momentum

$$\vec{m} = \frac{1}{2c^2}\mathrm{Re}\left[\vec{r}\times\left(\vec{E}\times\vec{H}^*\right)\right]$$

$$\vec{m} = \frac{\mu_0}{2\omega}\mathrm{Re}\left[\vec{H}^*\left(\vec{L}\cdot\vec{H}\right)\right]$$

$$\frac{d\vec{M}}{dr} = \frac{\mu_0}{2\omega k^2}\mathrm{Re}\sum_{mm'} a_E^*(l,m)\,a_E(l,m')\int Y_{lm'}^*\vec{L}Y_{lm}\,d\Omega$$

$$\frac{dM_z}{dr} = \frac{\mu_0}{2\omega k^2}\sum_m m\left|a_E(l,m)\right|^2$$

$$\frac{dM_z}{dr} = \frac{m}{\omega}\frac{dU}{dr}$$

photon interpretation: photon takes away

$m\hbar$ units of angular momentum

Angular Distribution

$$\vec{H} \to \frac{e^{ikr}}{kr}\frac{1}{2c^2}\sum_{lm}(-i)^{l+1}\left[a_E(l,m)\vec{X}_{lm} + a_M(l,m)\hat{n}\times\vec{X}_{lm}\right]$$

$$\vec{E} \to Z_0\vec{H}\times\hat{n}$$

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2}\left|\sum_{lm}(-i)^{l+1}\left[a_E(l,m)\vec{X}_{lm}\times\hat{n} + a_M(l,m)\vec{X}_{lm}\right]\right|^2$$

pure multipole

$$\frac{dP(l,m)}{d\Omega} = \frac{Z_0}{2k^2}\left|a(l,m)\right|^2\left|\vec{X}_{lm}\right|^2$$

$$= \frac{Z_0}{2k^2 l(l+1)}\left|a(l,m)\right|^2\left\{\begin{array}{c}\frac{1}{2}(l-m)(l+m+1)\left|Y_{l,m+1}\right|^2\\ +\frac{1}{2}(l+m)(l-m+1)\left|Y_{l,m-1}\right|^2 + m^2\left|Y_{l,m}\right|^2\end{array}\right\}$$

Sum Rule

$$\sum_{m=-l}^{l} \left| \vec{X}_{lm}(\theta,\phi) \right|^2 = \frac{2l+1}{4\pi}$$

$$P(l,m) = \frac{Z_0}{2k^2} \left| a(l,m) \right|^2$$

$$P = \frac{Z_0}{2k^2} \sum_{l,m} \left[\left| a_E(l,m) \right|^2 + \left| a_M(l,m) \right|^2 \right]$$