Physics 704/804 Electromagnetic Theory II

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Spherical Waves

$$\nabla^2\Psi + k^2\Psi = 0$$

$$\Psi = \sum_{l,m} f_{lm}(r) Y_{lm}(\theta,\phi)$$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}\right] f_l(r) = 0$$

$$f_l(r) = \frac{1}{r^{1/2}} u_l(r)$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 - \frac{(l+1/2)^2}{r^2}\right] f_l(r) = 0$$

solutions are Bessel Functions of order $l+1/2$

Spherical Bessel and Hankel Functions

$$f_{lm}(r) = \frac{A_{lm}}{r^{1/2}} J_{l+1/2}(kr) + \frac{B_{lm}}{r^{1/2}} N_{l+1/2}(kr)$$

$$j_l(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{l+1/2}(kr)$$

$$n_l(x) = \left(\frac{\pi}{2x}\right)^{1/2} N_{l+1/2}(kr)$$

$$h_l^{(1,2)} = \left(\frac{\pi}{2x}\right)\left[J_{l+1/2}(kr) \pm i N_{l+1/2}(kr)\right]$$

index 1 usual for the usual outgoing scattered wave

Some Examples

in general

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\sin x}{x}\right)$$

$$j_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\cos x}{x}\right)$$

$$j_0(x) = \frac{\sin x}{x}, \quad n_0(x) = \frac{\sin x}{x}, \quad h_0^1(x) = \frac{e^{ix}}{ix},$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x},$$

$$h_1(x) = -\frac{e^{ix}}{x}\left(1 - \frac{i}{x}\right)$$

Argument Limits

$$x << 1$$

$$j_l(x) \to \frac{x^l}{(2l+1)!!}\left(1+\frac{x^2}{2(2l+3)}+\cdots\right)$$

$$n_l(x) \to -\frac{(2l+1)!!}{x^{l+1}}\left(1+\frac{x^2}{2(1-2l)}+\cdots\right)$$

$$x >> 1$$

$$j_l(x) \to \frac{1}{x}\sin(x-l\pi/2)$$

$$j_l(x) \to -\frac{1}{x}\cos(x-l\pi/2)$$

$$h_l^1(x) \to (-i)^{l+1}\frac{e^{ix}}{x}$$

Green Function Expansion

$$\psi(\vec{x}) = \sum_{l,m} \left[A_{lm}^{1} h_l^{1}(kr) + A_{lm}^{2} h_l^{2}(kr) \right] Y_{lm}(\theta,\phi)$$

incoming and outgoing waves

$$\left(\nabla^2 + k^2\right) G(\vec{x},\vec{x}') = -\delta(\vec{x}-\vec{x}')$$

$$G(\vec{x},\vec{x}') = \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|}$$

$$G(\vec{x},\vec{x}') = \sum_{l,m} g_l(r,r') Y_{lm}^{*}(\theta',\phi') Y_{lm}(\theta,\phi)$$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}\right] g_l = -\frac{1}{r^2}\delta(r-r')$$

$$g_l(r,r') = A j_l(kr_<) h_l^{1}(kr_>)$$

Angular Momentum

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} = ik\sum_{l,m} j_l(kr_<)h_l^1(kr_>)Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$L^2Y_{lm}(\theta,\phi) = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{lm}$$

$$= l(l+1)Y_{lm} \qquad\qquad \vec{L} = \frac{1}{i}\left(\vec{r}\times\vec{\nabla}\right)$$

$$L_+ = L_x + iL_y$$

$$L_- = L_x - iL_y$$

$$L_z = -i\frac{\partial}{\partial\phi}$$

$$\vec{r}\cdot\vec{L} = 0$$

Applied to Spherical Harmonics

$$L_+ Y_{lm} = \sqrt{(l-m)(l+m+1)}\,Y_{lm}$$

$$L_- Y_{lm} = \sqrt{(l+m)(l-m+1)}\,Y_{lm}$$

$$L_z Y_{lm} = m Y_{lm}$$

$$L^2 \vec{L} = \vec{L} L^2$$

$$\vec{L} \times \vec{L} = i\vec{L}$$

$$L_j \nabla^2 = \nabla^2 L_j$$