Physics 704/804 Electromagnetic Theory II

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Far Field

exponential replaced with

$$\left|\vec{x} - \vec{x}'\right| \approx r - \hat{n} \cdot \vec{x}' \qquad\qquad \hat{n} = \vec{x} / \left|\vec{x}\right|$$

$$\lim_{kr \to \infty} \vec{A}\left(\vec{x}\right) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int \vec{J}\left(\vec{x}'\right) e^{-ik\hat{n}\cdot\vec{x}'} d^3\vec{x}'$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \sum_{n=0} \frac{\left(-ik\right)^n}{n!} \int \vec{J}\left(\vec{x}'\right)\left(\hat{n} \cdot \vec{x}'\right)^n d^3\vec{x}'$$

radiation pattern dominated by emission from first non-vanishing n

Magnetic Dipole/Electric Quadrupole Radiation

$n = 1$ term

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\int \vec{J}(\vec{x}')(\vec{n}\cdot\vec{x}')\,d^3x'$$

Integrand looks like

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \propto \int \begin{pmatrix} J_x\hat{n}_x & J_x\hat{n}_y & J_x\hat{n}_z \\ J_y\hat{n}_x & J_y\hat{n}_y & J_y\hat{n}_z \\ J_z\hat{n}_x & J_z\hat{n}_y & J_z\hat{n}_z \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} d^3\vec{x}'$$

symmetrize and antisymmetrize operator action on $\vec{J}, \vec{x}'$

$$\vec{J}\hat{n}\cdot\vec{x}' = \frac{1}{2}\left[\vec{J}\hat{n}\cdot\vec{x}' + \vec{x}'\hat{n}\cdot\vec{J}\right] + \frac{1}{2}\left[\vec{J}\hat{n}\cdot\vec{x}' - \vec{x}'\hat{n}\cdot\vec{J}\right] = \frac{1}{2}\left[\vec{J}\hat{n}\cdot\vec{x}' + \vec{x}'\hat{n}\cdot\vec{J}\right]$$

$$+\frac{1}{2}\left[\hat{n}\times\left(\vec{J}\times\vec{x}'\right)\right] = \frac{1}{2}\left[\vec{J}\hat{n}\cdot\vec{x}' + \vec{x}'\hat{n}\cdot\vec{J}\right] + \frac{1}{2}\left[\left(\vec{x}'\times\vec{J}\right)\times\hat{n}\right]$$

magnetization from current

$$\vec{M} = \frac{1}{2}\left(\vec{x} \times \vec{J}\right)$$

$$\vec{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}(\hat{n} \times \vec{m})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

magnetic dipole moment

$$\vec{m} = \int \vec{M} d^3\vec{x} = \frac{1}{2}\int \left(\vec{x} \times \vec{J}\right) d^3\vec{x}$$

comparing with the electric dipole results, $\vec{A}$ is identical, up to a constant factor, to the $\vec{H}$ for the electric dipole. Transcribe fields from electric dipole by replacement $\vec{p} \rightarrow \vec{m} / c$

Fields

$$\vec{H} = \frac{1}{4\pi}\left\{k^2(\hat{n}\times\vec{m})\times\hat{n}\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}\right]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}$$

$$\vec{E} = -\frac{Z_0 k^2}{4\pi}(\hat{n}\times\vec{m})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

radiation zone

$$\vec{H} = \frac{k^2}{4\pi}(\hat{n}\times\vec{m})\times\hat{n}\frac{e^{ikr}}{r}$$

$$\vec{E} = Z_0\vec{H}\times\hat{n}$$

because of the reversal of $\vec{E}$ and $\vec{H}$, the electric field is now perpendicular to the plane defined by $\hat{n}$ and $\vec{m}$.

Power

$$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}\left[r^2 \hat{n} \cdot \vec{E} \times \vec{H}^* \right]$$

$$\frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2} k^4 \left| \left(\hat{n} \times \vec{m} \right) \times \hat{n} \right|^2$$

if all components of $\vec{m}$ have the same phase

$$\frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2} k^4 \left| \vec{m} \right|^2 \sin^2 \theta$$

Electric Quadrupole Radiation

$$\frac{1}{2}\int\left[\left(\hat{n}\cdot\vec{x}'\right)\vec{J}+\left(\hat{n}\cdot\vec{J}\right)\vec{x}'\right]d^3\vec{x}'=\frac{-i\omega}{2}\int\vec{x}'\left(\hat{n}\cdot\vec{x}'\right)\rho\left(\vec{x}'\right)d^3\vec{x}'$$

$$\vec{A}\left(\vec{x}\right)=-\frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\int\vec{x}'\left(\hat{n}\cdot\vec{x}'\right)\rho\left(\vec{x}'\right)d^3\vec{x}'$$

in radiation zone the expressions become

$$\vec{H}\approx ik\hat{n}\times\vec{A}/\mu_0=-\frac{ick^3}{8\pi}\frac{e^{ikr}}{r}\int\hat{n}\times\vec{x}'\left(\hat{n}\cdot\vec{x}'\right)\rho\left(\vec{x}'\right)d^3\vec{x}'$$

$$\vec{E}\approx ikZ_0\left(\hat{n}\times\vec{A}\right)\times\hat{n}/\mu_0$$

Recall quadrupole moment tensor

$$Q_{ab}=\int\left(3x_a x_b-r^2\delta_{ab}\right)\rho\left(\vec{x}\right)d^3\vec{x}$$

$$\hat{n}\times\int\vec{x}'\left(\hat{n}\cdot\vec{x}'\right)\rho\left(\vec{x}'\right)d^3\vec{x}'=\frac{1}{3}\hat{n}\times\left(Q\cdot\hat{n}\right)\qquad\left(Q\cdot\hat{n}\right)_a\equiv\sum_b Q_{ab}\hat{n}_b$$

Radiation Pattern

$$\vec{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{n}\times\left(Q\cdot\hat{n}\right)$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2}k^6\left|\left(\hat{n}\times\left(Q\cdot\hat{n}\right)\right)\times\hat{n}\right|^2$$

$$\left|\left(\hat{n}\times\left(Q\cdot\hat{n}\right)\right)\times\hat{n}\right|^2 = Q^*\cdot Q - \left|\hat{n}\cdot Q\right|^2$$

angular integrals involve

$$\int n_a n_b \, d\Omega = \frac{4\pi}{3}\delta_{ab}$$

$$\int n_a n_b n_c n_d \, d\Omega = \frac{4\pi}{15}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)$$

$$P = \frac{c^2 Z_0 k^6}{1440}\sum_{a,b}\left|Q_{ab}\right|^2$$

Oscillating Spheroid

$$Q_{33} = Q_{0,} \qquad Q_{11} = Q_{22} = -\frac{1}{2} Q_0$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{512\pi^2} Q_0^2 \sin^2\theta \cos^2\theta$$

$$P = \frac{c^2 Z_0 k^6 Q_0^2}{960\pi}$$

Spherical Waves

$$\nabla^2\Psi + k^2\Psi = 0$$

$$\Psi = \sum_{l,m} f_{lm}(r)Y_{lm}(\theta,\phi)$$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{l(l+1)}{r^2}\right] f_l(r) = 0$$

$$f_l(r) = \frac{1}{r^{1/2}} u_l(r)$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 - \frac{(l+1/2)^2}{r^2}\right] f_l(r) = 0$$

solutions are Bessel Functions of order $l+1/2$