Physics 704/804 Electromagnetic Theory II

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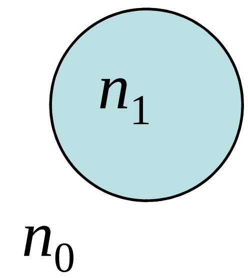
Optical Fibers

$$\Delta = \frac{n_1^2 - n_0^2}{2n_1^2} \approx 1 - \frac{n_0}{n_1} \qquad \theta_{\max} = \cos^{-1}\left(n_0 / n_1\right)$$



$$dN = \pi a^2 \frac{d^2k}{\left(2\pi\right)^2} 2$$

$$N \approx a^2 k^2 \int_0^{\theta_{\max}} \theta \, d\theta = \frac{1}{2}\left(ka\sqrt{2\Delta}\right)^2 = \frac{1}{2} V^2$$

fibre parameter $\qquad V = ka\sqrt{2\Delta}$

$\lambda = 0.85\text{mm},\ n_1 = 1.4,\ a = 25 \text{ micron}$

$N = 335, \quad$ many modes excited

Field Equations

$$n\left(\vec{x}\right)=\sqrt{\varepsilon\left(\vec{x}\right)/\varepsilon_0}$$

$$\nabla^2\vec{E}+\mu_0\varepsilon\left(\vec{x}\right)\omega^2\vec{E}+\nabla\left(\frac{1}{\varepsilon}\vec{E}\cdot\nabla\varepsilon\right)=0$$

$$\nabla^2\vec{H}+\mu_0\varepsilon\left(\vec{x}\right)\omega^2\vec{H}-i\omega\nabla\varepsilon\times\vec{E}$$

$$\nabla^2\psi+\frac{\omega^2}{c^2}n^2\left(\vec{x}\right)\psi=0$$

JWKB (eikonal) approximation

$$\psi\propto e^{i\omega S\left(\vec{x}\right)/c}$$

$$\frac{\omega^2}{c^2}\left[\vec{\nabla}S\cdot\vec{\nabla}S-n^2\left(\vec{x}\right)\right]+i\frac{\omega}{c}\nabla^2S=0$$

Generalized Snell's Law

$$\vec{\nabla} S \cdot \vec{\nabla} S = \frac{\omega^2}{c^2} n^2(\vec{x})$$

$$S(\vec{x}) = S(\vec{x}_0) + (\vec{x} - \vec{x}_0) \cdot \vec{\nabla} S(\vec{x}_0) + \cdots$$

$$\psi(\vec{x}) = \exp\left[i\omega S(\vec{x}_0)/c\right] \exp\left[i(\vec{x} - \vec{x}_0) \cdot \frac{\omega \vec{\nabla} S}{c}\right]$$

plane wave with

$$\vec{k}(\vec{x}_0) = \omega \vec{\nabla} S(\vec{x}_0)/c = n(\vec{x}_0)\hat{k}(\vec{x}_0)\omega/c$$

$$n(\vec{x})\frac{d\vec{r}}{ds} = \nabla S \qquad \frac{d}{ds}\left[n(\vec{x})\frac{d\vec{r}}{ds}\right] = \frac{d}{ds}\nabla S = \nabla \frac{dS}{ds}$$

$$\frac{d}{ds}\left[n(\vec{r})\frac{d\vec{r}}{ds}\right] = \nabla n(\vec{r})$$

Meridonal Ray Solutions

$$\frac{d}{ds}\left[n(x)\sin\theta(x)\right]=\frac{dn}{dx} \qquad \frac{d}{ds}\left[n(x)\cos\theta(x)\right]=0$$

$\bar{n}=n(x_{\max})$ is index of refraction at turning point

$$\frac{\bar{n}}{n(x)}\frac{d}{dz}\left(\bar{n}\frac{dx}{dz}\right)=\frac{dn}{dx} \qquad \bar{n}^2\frac{d^2x}{dz^2}=n(x)\frac{dn}{dx}$$

$$\bar{n}^2\left(\frac{dx}{dz}\right)^2=n^2(x)-\bar{n}^2 \qquad z(x)=\bar{n}\int_0^x\frac{dx}{\sqrt{n^2(x)-\bar{n}^2}}$$

1/2 period distance $Z=2\bar{n}\int_0^{x_{\max}}\frac{dx}{\sqrt{n^2(x)-\bar{n}^2}}$

Physical and Optical Paths

$$L_{phy} = \int_A^B ds \qquad L_{opt} = \int_A^B n(x)\, ds$$

for a half period

$$L_{phy} = \int_0^{x_{\max}} \frac{n(x)\, dx}{\sqrt{n^2(x) - \overline{n}^2}} \qquad L_{opt} = \int_0^{x_{\max}} \frac{n^2(x)\, dx}{\sqrt{n^2(x) - \overline{n}^2}}$$

Transit time for fiber with $z \gg Z$

$$T = \frac{L_{opt}}{Z} \frac{z}{c}$$

Radiating Systems; Multipoles

$$\rho(\vec{x},t) = \rho(\vec{x})e^{-i\omega t}$$

$$\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$$

following our standard practice

quantities are real parts

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\int dt' \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\delta\left(t' + |\vec{x}-\vec{x}'|/c - t\right)$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\vec{J}(\vec{x}')\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \qquad k = \omega / c$$

Fields

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}$$

$$\vec{E} = \frac{iZ_0}{k}\vec{\nabla}\times\vec{H} \qquad Z_0 = \sqrt{\mu_0/\varepsilon_0}$$

near field: $\quad d \ll r \ll \lambda = 2\pi c/\omega$

far field: $\quad d \ll \lambda \ll r$

Near Field

exponential replaced by 1

$$\frac{1}{4\pi\left|\vec{x}-\vec{x}'\right|}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right)$$

$$\lim_{\substack{kr\to 0,\\ d<<r}}\vec{A}\left(\vec{x}\right)=\mu_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{Y_{lm}\left(\theta,\phi\right)}{r^{l+1}}\int Y_{lm}^*\left(\theta',\phi'\right)r'^l\,\vec{J}\left(\vec{x}'\right)d^3\vec{x}'$$

Far Field

exponential replaced with

$$\left|\vec{x}-\vec{x}'\right| \approx r-\hat{n}\cdot\vec{x}' \qquad\qquad \hat{n}=\vec{x}/\left|\vec{x}\right|$$

$$\lim_{kr\to\infty}\vec{A}\left(\vec{x}\right)=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int\vec{J}\left(\vec{x}'\right)e^{-ik\hat{n}\cdot\vec{x}'}d^3\vec{x}'$$

$$=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{n=0}\frac{\left(-ik\right)^n}{n!}\int\vec{J}\left(\vec{x}'\right)\left(\hat{n}\cdot\vec{x}'\right)^n d^3\vec{x}'$$

radiation pattern dominated by emission from first non-vanishing n