# Physics 704/804 Electromagnetic Theory II

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# Boundary Conditions

$$\psi\big|_S = 0 \qquad (\mathrm{TM}) \qquad \left.\frac{\partial \psi}{\partial n}\right|_S = 0 \qquad (\mathrm{TE})$$

Spectrum of eigenvalues and eigenfunctions

$$\gamma_\lambda \qquad\qquad \psi_\lambda(x, y)$$

Wavelength in mode $\lambda$

$$k_\lambda^2 = \mu\varepsilon\omega^2 - \gamma_\lambda^2$$

# Cutoff Frequency

$$\omega_\lambda^2 = \frac{\gamma_\lambda^2}{\mu\varepsilon}$$

No propagation at frequencies below cutoff

$$k_\lambda^2 = \mu\varepsilon\left(\omega^2 - \omega_\lambda^2\right)$$

For single-mode propagation choose frequency to be above cutoff for lowest mode and below cutoff for all other modes. Phase velocity infinite at cutoff!

# Rectangular Waveguide

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\gamma^2\right)\psi = 0$$

$$\left.\frac{\partial\psi}{\partial n}\right|_S = 0 \text{ (TE)} \rightarrow \psi_{mn}(x,y) = A\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi x}{a}\right)$$

$$\gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu\varepsilon}}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{1/2}$$

# Lowest Mode

$$\omega_{10} = \frac{\pi}{\sqrt{\mu\varepsilon}}\frac{m}{a}$$

$$H_z = A\cos\left(\frac{\pi x}{a}\right)e^{ikz-i\omega t}$$

$$H_z = -\frac{ika}{\pi}A\sin\left(\frac{\pi x}{a}\right)e^{ikz-i\omega t}$$

$$E_y = i\frac{\omega a\mu}{\pi}A\sin\left(\frac{\pi x}{a}\right)e^{ikz-i\omega t}$$

Lowest TM mode has *m* and *n* one with a sin solution. Why?
Its cutoff for the lowest mode is at a frequency higher by

$$\left(1 + a^2 / b^2\right)^{1/2}$$

# Resonant Cavities

Put end conductors on a cylindrical waveguide. Example: Cylindrical cavity of length *d* and radius *R.* In general, *z* dependence is

$$A \sin(kz) + B\cos(kz)$$

$$BCs \rightarrow k = p\frac{\pi}{d}, \quad p = 0,1,2,\ldots$$

TM

$$E_z = \psi(x,y)\cos\left(\frac{p\pi}{d}\right), \quad p = 0,1,2,\ldots$$

TE

$$H_z = \psi(x,y)\sin\left(\frac{p\pi}{d}\right), \quad p = 1,2,\ldots$$

# Field Patterns

TM

$$\vec{E}_t = -\frac{p\pi}{d\gamma^2}\sin\left(\frac{p\pi}{d}\right)\vec{\nabla}_t\psi$$

$$\vec{H}_t = \frac{i\varepsilon\omega}{\gamma^2}\cos\left(\frac{p\pi}{d}\right)\hat{z}\times\vec{\nabla}_t\psi$$

TE

$$\vec{E}_t = -\frac{i\omega\mu}{\gamma^2}\sin\left(\frac{p\pi}{d}\right)\hat{z}\times\vec{\nabla}_t\psi$$

$$\vec{H}_t = \frac{p\pi}{d\gamma^2}\cos\left(\frac{p\pi}{d}\right)\vec{\nabla}_t\psi$$

$$\gamma^2 = \mu\varepsilon\omega^2 - \left(p\pi / d\right)^2$$

# Eigenvalue Equation

$$\omega_{\lambda p}^2 = \frac{1}{\mu\varepsilon}\left(\gamma_\lambda^2 + \left(p\pi / d\right)^2\right)$$

TM

$$\psi\left(\rho,\phi\right) = AJ_m\left(\gamma_{mn}\rho\right)e^{\pm im\phi}$$

$$\gamma_{mn} = \frac{x_{mn}}{R} \qquad J_m\left(x_{mn}\right) = 0$$

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2\pi^2}{d^2}} \qquad \omega_{010} = \frac{2.405}{\sqrt{\mu\varepsilon}R}$$

$$E_z = AJ_0\left(\frac{2.405\rho}{R}\right)e^{-i\omega t} \qquad H_\phi = -i\sqrt{\frac{\varepsilon}{\mu}}AJ_1\left(\frac{2.405\rho}{R}\right)e^{-i\omega t}$$

# TE

$$\psi(\rho,\phi) = AJ_m(\gamma_{mn}\rho)e^{\pm im\phi}$$

$$\gamma_{mn} = \frac{x'_{mn}}{R} \qquad J'_m(x'_{mn}) = 0$$

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{x'^2_{mn}}{R^2} + \frac{p^2\pi^2}{d^2}}$$

$$\omega_{111} = \frac{1.841}{\sqrt{\mu\varepsilon}R}\left(1 + 2.912\frac{R^2}{d^2}\right)$$

$$H_z = AJ_1\left(\frac{1.841\rho}{R}\right)\cos\phi\sin\left(\frac{\pi z}{d}\right)e^{-i\omega t}$$