Physics 704/804 Electromagnetic Theory II

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

Perpendicular Polarization

$$0 = \left(\left|\vec{E}_0\right| + \left|\vec{E}_0''\right| - \left|\vec{E}_0'\right|\right) \qquad \text{Tangential } \vec{E}$$

$$0 = \sqrt{\frac{\varepsilon}{\mu}}\left(\left|\vec{E}_0\right| - \left|\vec{E}_0''\right|\right)\cos i - \sqrt{\frac{\varepsilon'}{\mu'}}\left|\vec{E}_0'\right|\cos r \qquad \text{Tangential } \vec{H}$$

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

Parallel Polarization

$$0 = \cos i\left(\left|\vec{E}_0\right| - \left|\vec{E}_0''\right|\right) - \cos r\left|\vec{E}_0'\right| \qquad \text{Tangential } \vec{E}$$

$$0 = \sqrt{\frac{\varepsilon}{\mu}}\left(\left|\vec{E}_0\right| + \left|\vec{E}_0''\right|\right) - \sqrt{\frac{\varepsilon'}{\mu'}}\left|\vec{E}_0'\right| \qquad \text{Tangential } \vec{H}$$

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{\frac{\mu}{\mu'}n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}$$

Mistakes from Last Time

Energy Conservation at Normal Incidence

$$\sqrt{\frac{\varepsilon}{\mu}}\frac{E_0^2}{2} = \sqrt{\frac{\varepsilon}{\mu}}\frac{E_0''^2}{2} + \sqrt{\frac{\varepsilon'}{\mu'}}\frac{E_0'^2}{2}$$

$$\frac{\left|\vec{E}_0'\right|}{\left|\vec{E}_0\right|} = \frac{2}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} = \frac{2n}{n'+n}$$

$$\frac{\left|\vec{E}_0''\right|}{\left|\vec{E}_0\right|} = \frac{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}-1}{\sqrt{\frac{\mu\varepsilon'}{\mu'\varepsilon}}+1} = \frac{n'-n}{n'+n}$$

Brewster's Angle Calculation

$$n'^2 \cos i_B = n\sqrt{n'^2 - n^2 \sin^2 i_B}$$

$$n'^4 \cos^2 i_B = n^2 \left(n'^2 - n^2 \sin^2 i_B\right)$$

$$n'^2 = \frac{+n^2 \pm \sqrt{n^4 - 4n^4 \cos^2 i_B \sin^2 i_B}}{2\cos^2 i_B} = \frac{+n^2 \pm n^2\left(1 - 2\cos^2 i_B\right)}{2\cos^2 i_B}$$

$$i_{Brewster} = \tan^{-1}\frac{n'}{n}$$

Zero reflection in perpendicular polarization implies $n' = n$

Dispersion

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(k) e^{ikx-\omega(k)t} dk$$

Have exact calculation for modulated Gaussian function

$$u(x,0) = \exp\left(-x^2/2L^2\right) e^{ik_0 x}$$

$$A(k) = \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx$$

$$= \sqrt{2\pi} L \exp\left(-L^2/2\right)\left(k-k_0\right)^2$$

$$\omega(k) = \nu\left(1+\frac{a^2k^2}{2}\right)$$

Pulse Spreading, or Dispersion

$$\mathrm{v}_g = \frac{d\omega}{dk} = \nu a^2 k_0$$

$$u(x,t) = \frac{L}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-L^2/2\right)\left(k-k_0\right)^2} e^{ikx - i\nu t\left[1+\left(a^2k^2/2\right)\right]} dk$$

$$= \frac{\exp\left[-\dfrac{\left(x-\nu a^2 k_0 t\right)^2}{2L^2\left(1+\dfrac{ia^2\nu t}{L^2}\right)}\right]}{\left(1+\dfrac{ia^2\nu t}{L^2}\right)^{1/2}} \exp\left[ik_0 x - i\nu\left(1+a^2k^2/2\right)t\right]$$

$$L(t) = \frac{d\omega}{dk} = \sqrt{L^2 + \left(\nu a^2 t / L\right)^2}$$

$$\Delta \mathrm{v}_g = \frac{d^2\omega}{dk^2}\Delta k = \frac{\nu a^2}{L}$$

$$\Delta x(t) = \sqrt{\left(\Delta x\right)^2 + \left(\Delta \mathrm{v}_g t\right)^2}$$

Causality

$$\vec{D}(\vec{x},\omega) = \varepsilon(\omega)\vec{E}(\vec{x},\omega)$$

Convolution Theorem (Fahltung Theorem) implies non-locality in time.

$$\vec{D}(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\vec{D}(\vec{x},\omega)e^{-i\omega t}d\omega = \int_{-\infty}^{\infty}\varepsilon(\omega)\vec{E}(\vec{x},\omega)e^{-i\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon(\omega)e^{-i\omega t}\int_{\infty}^{\infty}dt'e^{+i\omega t'}\vec{E}(\vec{x},t')d\omega$$

$$= \varepsilon_0\left\{\vec{E}(\vec{x},t) + \int_{-\infty}^{\infty}G(\tau)\vec{E}(\vec{x},t-\tau)d\tau\right\}$$

Green function for connection

$$G(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\varepsilon(\omega)/\varepsilon_0 - 1\right]e^{-i\omega\tau}\,d\omega$$

Damped oscillator connection

$$\varepsilon(\omega)/\varepsilon_0 - 1 = \omega_p^2\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)^{-1}$$

$$G(\tau) = \omega_p^2 e^{-\gamma t/2}\frac{\sin\nu_0\tau}{\nu_0}\Theta(\tau)$$

$$\nu_0 = \sqrt{\omega_0^2 - \gamma^2/4}$$

Vanishes for negative τ, cause cannot precede effect. Causal Green's functions must be analytic in upper ½ of complex plane.

Kramers-Kronig Relations

$$\varepsilon(\omega)/\varepsilon_0 - 1 = \int_0^\infty G(\tau) e^{i\omega\tau} d\tau$$

Is automatically causal for a wide variety of choices for *G*. Analyticity in UH-ωP implies a relationship between real and imaginary part of the permitivity. Cauchy's theorem for *z* inside a closed curve *C*

$$\varepsilon(z)/\varepsilon_0 = 1 + \frac{1}{2\pi i} \oint_C \frac{\varepsilon(\omega')/\varepsilon_0 - 1}{\omega' - z} d\omega'$$

$$= 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varepsilon(\omega')/\varepsilon_0 - 1}{\omega' - z} d\omega'$$

where the integral is now along the real axis

$$\frac{1}{\omega-\omega'-i\delta}=P\left(\frac{1}{\omega-\omega'}\right)+\pi i\delta\left(\omega'-\omega\right)$$

$$\mathrm{Re}\,\varepsilon\left(\omega\right)/\varepsilon_0=1+\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\,\varepsilon\left(\omega'\right)/\varepsilon_0}{\omega'-z}d\omega'$$

$$\mathrm{Im}\,\varepsilon\left(\omega\right)/\varepsilon_0=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Re}\,\varepsilon\left(\omega'\right)/\varepsilon_0-1}{\omega'-z}d\omega'$$

$$\mathrm{Re}\,\varepsilon\left(\omega\right)/\varepsilon_0=1+\frac{2}{\pi}P\int_{0}^{\infty}\frac{\omega'\,\mathrm{Im}\,\varepsilon\left(\omega'\right)/\varepsilon_0}{\omega'^2-\omega^2}d\omega'$$

$$\mathrm{Im}\,\varepsilon\left(\omega\right)/\varepsilon_0=-\frac{2\omega}{\pi}P\int_{0}^{\infty}\frac{\mathrm{Re}\,\varepsilon\left(\omega'\right)/\varepsilon_0-1}{\omega'^2-\omega^2}d\omega'$$

$$\varepsilon\left(-\omega\right)=\varepsilon^*\left(\omega^*\right)$$

Sum Rules

Sum Rules for oscillator strengths

$$\omega_p^2 = 1 + \frac{2}{\pi} P \int_0^\infty \omega \,\mathrm{Im}\, \varepsilon(\omega) / \varepsilon_0 d\omega$$

Second Sum Rule

$$\frac{1}{N} \int_0^N \mathrm{Re}\, \varepsilon(\omega) / \varepsilon_0 d\omega = 1 + \frac{\omega_p^2}{N^2}$$