Physics 704/804 Electromagnetic Theory II

G. A. Krafft

Jefferson Lab

Jefferson Lab Professor of Physics

Old Dominion University

Energy Conservation

Energy/volume involves (exterior) multiplication of *E* and *D*. The exact formula comes from the Maxwell Equations:

$$\int_V \omega_E^1 \wedge \omega_J^2 = \int_V \omega_E^1 \wedge \left(d\,\omega_H^1 - \frac{\partial}{\partial t}\omega_D^2 \right) - \int_V \left(d\,\omega_E^1 + \frac{\partial}{\partial t}\omega_B^2 \right) \wedge \omega_H^1$$

$$= -\int_V d\left(\omega_E^1 \wedge \omega_H^1\right) - \int_V \left[\omega_E^1 \wedge \frac{\partial}{\partial t}\omega_D^2 + \omega_H^1 \wedge \frac{\partial}{\partial t}\omega_B^2 \right]$$

The RHS terms define the [1] Poynting (Energy Flux) form

$$\omega_E^1 \wedge \omega_H^1 = \omega_{\vec{E}\times\vec{H}}^2$$

For linear materials have [2] energy density form

$$\frac{1}{2}\omega_E^1 \wedge \omega_D^2 + \frac{1}{2}\omega_H^1 \wedge \omega_B^2 = \left(\frac{\vec{E}\cdot\vec{D}}{2} + \frac{\vec{H}\cdot\vec{B}}{2} \right) dx \wedge dy \wedge dz$$

In Vector Form

Energy Density (J/m^3)

$$u = \frac{\vec{E}\cdot\vec{D}}{2} + \frac{\vec{H}\cdot\vec{B}}{2}$$

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E}$$

Poynting Vector (J/sec m^2)

$$\vec{S} = \vec{E}\times\vec{H}$$

Momentum Conservation

Momentum is a vector quantity. To describe momentum densities need (what could be more natural) vector valued forms. Mechanical momentum delivered to a load by an electromagnetic field is

$$\frac{d\vec{p}_{mech}}{dt} = \int_V \left[\rho \vec{E} + \vec{J} \times \vec{B} \right] dxdydx$$

Define the vector 3-form force density to be

$$\left[\rho \vec{E} + \vec{J} \times \vec{B} \right] dx \wedge dy \wedge dz$$

Momentum Flux density must have nine components

$$\hat{e}_i T_{ij} \varepsilon_{jkl} dx^k \otimes dx^l = \hat{e}_i \left[T_{i1} dy \wedge dz + T_{i2} dz \wedge dx + T_{i3} dx \wedge dy \right]$$

In free space

$$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E} \qquad \vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

then

$$\rho \vec{E} + \vec{J} \times \vec{B} = \varepsilon_0 \begin{bmatrix} \vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) + c^2 \vec{B}\left(\vec{\nabla} \cdot \vec{B}\right) \\ -\vec{E} \times \left(\vec{\nabla} \times \vec{E}\right) - c^2 \vec{B} \times \left(\vec{\nabla} \times \vec{B}\right) \end{bmatrix} - \varepsilon_0 \frac{\partial}{\partial t}\left(\vec{E} \times \vec{B}\right)$$

Momentum change equation

$$\frac{d\vec{p}_{mech}}{dt} + \frac{d\vec{p}_{field}}{dt} =$$

$$\varepsilon_0 \int_V \left[\vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) + c^2 \vec{B}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{E} \times \left(\vec{\nabla} \times \vec{E}\right) - c^2 \vec{B} \times \left(\vec{\nabla} \times \vec{B}\right) \right] dx \wedge dy \wedge dz$$

Maxwell Stress Tensor

Momentum in field

$$\vec{p}_{field} = \frac{1}{c^2}\int_V \vec{E}\times\vec{H}\,dxdydz$$

Stress Tensor

$$T_{ij} = \varepsilon_0\left[E_iE_j + c^2B_iB_j - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right)\delta_{ij}\right] =$$

$$\varepsilon_0\begin{pmatrix} E_xE_x + c^2B_xB_x & E_xE_y + c^2B_xB_y & E_xE_z + c^2B_xB_z \\ E_yE_x + c^2B_yB_x & E_yE_y + c^2B_yB_y & E_yE_z + c^2B_yB_z \\ E_zE_x + c^2B_zB_x & E_zE_y + c^2B_zB_y & E_zE_z + c^2B_zB_z \end{pmatrix} - \frac{\varepsilon_0}{2}\begin{pmatrix} \vec{E}\cdot\vec{E} + \\ c^2\vec{B}\cdot\vec{B} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Harmonic Poynting Theorem

Represent the (real) electromagnetic field as

$$\vec{E}(\vec{x},t) = \mathrm{Re}\left(\vec{E}(\vec{x})e^{-i\omega t}\right) = \frac{1}{2}\left[\vec{E}(\vec{x})e^{-i\omega t} + \vec{E}^*(\vec{x})e^{i\omega t}\right]$$

$$\vec{J}(\vec{x},t)\cdot\vec{E}(\vec{x},t) = \frac{1}{2}\mathrm{Re}\left[\vec{J}^*(\vec{x})\cdot\vec{E}(\vec{x}) + \vec{J}(\vec{x})\cdot\vec{E}(\vec{x})e^{i2\omega t}\right]$$

$$\left\langle \vec{J}(\vec{x},t)\cdot\vec{E}(\vec{x},t)\right\rangle_{time\ average} = \frac{1}{2}\mathrm{Re}\left[\vec{J}^*(\vec{x})\cdot\vec{E}(\vec{x})\right]$$

then define the complex Poynting vector

$$\vec{S} = \frac{1}{2}\left(\vec{E}\times\vec{H}^*\right)$$

and the complex electric and magnetic densities

$$w_{el} = \frac{1}{4}\vec{E}\left(\vec{x}\right)\cdot\vec{D}^*\left(\vec{x}\right), \qquad w_{ma} = \frac{1}{4}\vec{B}\left(\vec{x}\right)\cdot\vec{H}^*\left(\vec{x}\right)$$

Real part of following equation gives energy conservation

$$\frac{1}{2}\int_V \left(\vec{J}^*\cdot E\right) d^3x + 2i\omega\int_V \left(w_{el} - w_{ma}\right) d^3x + \oint_S \vec{S}\cdot n\,da = 0$$

Transformation of E-M Fields

One of the limitations of conventional vector analysis is that scalars and pseudoscalars, vectors and axial vectors, etc., are not distinguished. This distinction is explicit in the form language, and makes it straightforward to determine the transformation properties of different field quantities.

Ordinary vector:

$$x' = -x,\, y' = -y,\, z' = -z$$

$$\omega'^{1}_{E'} = \omega^{1}_{E} \rightarrow \vec{E}'(-\vec{x},t) = -\vec{E}(\vec{x},t)$$

Axial vector:

$$x' = -x,\, y' = -y,\, z' = -z$$

$$\omega'^{2}_{B'} = \omega^{2}_{B} \rightarrow \vec{B}'(-\vec{x},t) = \vec{B}(\vec{x},t)$$

Plane Electromagnetic Waves

If there are no sources in the Maxwell equations we must solve

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0$$

For a harmonic time dependence $\vec{B}, \vec{E} \propto e^{-i\omega t}$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} - i\omega \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} + i\omega\mu\varepsilon \vec{E} = 0$$

Get

$$\left[\nabla^2 + \mu\varepsilon\omega^2\right]\vec{E} = 0 \rightarrow \vec{E} = \vec{E}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$\left[\nabla^2 + \mu\varepsilon\omega^2\right]\vec{B} = 0 \rightarrow \vec{B} = \vec{B}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad \left|\vec{k}\right| = \sqrt{\mu\varepsilon}\,\omega$$

Divergence equations imply

$$\vec{k}\cdot\vec{E}_0 = \vec{k}\cdot\vec{B}_0 = 0$$

Faraday's Law implies

$$\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega} = \frac{n\hat{n}\times\vec{E}_0}{c} \qquad \hat{n} = \frac{\vec{k}}{\left|\vec{k}\right|}$$

where *n* is the index of refraction

Energy and Energy Flux

Poynting vector for the solution is

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{n}{2\mu c}\left|\vec{E}_0\right|^2 \hat{n} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left|\vec{E}_0\right|^2 \hat{n}$$

Energy Density

$$u = \frac{1}{4}\left[\varepsilon\vec{E}\cdot\vec{E}^* + \frac{1}{\mu}\vec{B}\cdot\vec{B}^*\right] = \frac{\varepsilon}{2}\left|\vec{E}_0\right|^2$$

Consistency

$$\left|\vec{S}\right| = cu / n$$

Note that the units are correct