Physics 704/804 Electromagnetic Theory II

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Momentum Conservation

Momentum is a vector quantity. To describe momentum densities need (what could be more natural) vector valued forms. Mechanical momentum delivered to a load by an electromagnetic field is

$$\frac{d\vec{p}_{mech}}{dt} = \int_V \left[\rho \vec{E} + \vec{J} \times \vec{B} \right] dxdydx$$

Define the vector 3-form force density to be

$$\left[\rho \vec{E} + \vec{J} \times \vec{B} \right] dx \wedge dy \wedge dz$$

Momentum Flux density must have nine components

$$\hat{e}_i T_{ij} \varepsilon_{jkl} dx^k \otimes dx^l = \hat{e}_i \left[T_{i1} dy \wedge dz + T_{i2} dz \wedge dx + T_{i3} dx \wedge dy \right]$$

In free space

$$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E} \qquad \vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

then

$$\rho \vec{E} + \vec{J} \times \vec{B} = \varepsilon_0 \begin{bmatrix} \vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) + c^2 \vec{B}\left(\vec{\nabla} \cdot \vec{B}\right) \\ -\vec{E} \times \left(\vec{\nabla} \times \vec{E}\right) - c^2 \vec{B} \times \left(\vec{\nabla} \times \vec{B}\right) \end{bmatrix} - \varepsilon_0 \frac{\partial}{\partial t}\left(\vec{E} \times \vec{B}\right)$$

Momentum change equation

$$\frac{d\vec{p}_{mech}}{dt} + \frac{d\vec{p}_{field}}{dt} =$$

$$\varepsilon_0 \int_V \left[\vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) + c^2 \vec{B}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{E} \times \left(\vec{\nabla} \times \vec{E}\right) - c^2 \vec{B} \times \left(\vec{\nabla} \times \vec{B}\right) \right] dx \wedge dy \wedge dz$$

Maxwell Stress Tensor

Momentum in field

$$\vec{p}_{field} = \frac{1}{c^2}\int_V \vec{E} \times \vec{H} dxdydz$$

Stress Tensor

$$T_{ij} = \varepsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2 \vec{B}\cdot\vec{B}\right)\delta_{ij} \right] =$$

$$\varepsilon_0 \begin{pmatrix} E_x E_x + c^2 B_x B_x & E_x E_y + c^2 B_x B_y & E_x E_z + c^2 B_x B_z \\ E_y E_x + c^2 B_y B_x & E_y E_y + c^2 B_y B_y & E_y E_z + c^2 B_y B_z \\ E_z E_x + c^2 B_z B_x & E_z E_y + c^2 B_z B_y & E_z E_z + c^2 B_z B_z \end{pmatrix} - \frac{\varepsilon_0}{2}\begin{pmatrix} \vec{E}\cdot\vec{E} + \\ c^2 \vec{B}\cdot\vec{B} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Harmonic Poynting Theorem

Represent the (real) electromagnetic field as

$$\vec{E}\left(\vec{x},t\right)=\mathrm{Re}\left(\vec{E}\left(\vec{x}\right)e^{-i\omega t}\right)=\frac{1}{2}\left[\vec{E}\left(\vec{x}\right)e^{-i\omega t}+\vec{E}^{*}\left(\vec{x}\right)e^{i\omega t}\right]$$

$$\vec{J}\left(\vec{x},t\right)\cdot\vec{E}\left(\vec{x},t\right)=\frac{1}{2}\mathrm{Re}\left[\vec{J}^{*}\left(\vec{x}\right)\cdot\vec{E}\left(\vec{x}\right)+\vec{J}\left(\vec{x}\right)\cdot\vec{E}\left(\vec{x}\right)e^{i2\omega t}\right]$$

$$\left\langle\vec{J}\left(\vec{x},t\right)\cdot\vec{E}\left(\vec{x},t\right)\right\rangle_{time\ average}=\frac{1}{2}\mathrm{Re}\left[\vec{J}^{*}\left(\vec{x}\right)\cdot\vec{E}\left(\vec{x}\right)\right]$$

then define the complex Poynting vector

$$\vec{S}=\frac{1}{2}\left(\vec{E}\times\vec{H}^{*}\right)$$

and the complex electric and magnetic densities

$$w_{el} = \frac{1}{4}\vec{E}\left(\vec{x}\right)\cdot\vec{D}^{*}\left(\vec{x}\right), \qquad w_{ma} = \frac{1}{4}\vec{B}\left(\vec{x}\right)\cdot\vec{H}^{*}\left(\vec{x}\right)$$

Real part of following equation gives energy conservation

$$\frac{1}{2}\int_V \left(\vec{J}^{*}\cdot E\right)d^3x + 2i\omega\int_V \left(w_{el} - w_{ma}\right)d^3x + \oint_S \vec{S}\cdot n\,da = 0$$

Transformation of E-M Fields

One of the limitations of conventional vector analysis is that scalars and pseudoscalars, vectors and axial vectors, etc., are not distinguished. This distinction is explicit in the form language, and makes it straightforward to determine the transformation properties of different field quantities.

Ordinary vector:

$$x' = -x,\ y' = -y,\ z' = -z$$

$$\omega'^{1}_{E'} = \omega^{1}_{E} \rightarrow \vec{E}'\left(-\vec{x},t\right) = -\vec{E}\left(\vec{x},t\right)$$

Axial vector:

$$x' = -x,\ y' = -y,\ z' = -z$$

$$\omega'^{2}_{B'} = \omega^{2}_{B} \rightarrow \vec{B}'\left(-\vec{x},t\right) = \vec{B}\left(\vec{x},t\right)$$

Plane Electromagnetic Waves

If there are no sources in the Maxwell equations we must solve

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0$$

For a harmonic time dependence $\vec{B}, \vec{E} \propto e^{-i\omega t}$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} - i\omega \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} + i\omega\mu\varepsilon \vec{E} = 0$$

Get

$$\left[\nabla^2 + \mu\varepsilon\omega^2\right]\vec{E} = 0 \rightarrow \vec{E} = \vec{E}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$\left[\nabla^2 + \mu\varepsilon\omega^2\right]\vec{B} = 0 \rightarrow \vec{B} = \vec{B}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad \left|\vec{k}\right| = \sqrt{\mu\varepsilon}\,\omega$$

Divergence equations imply

$$\vec{k}\cdot\vec{E}_0 = \vec{k}\cdot\vec{B}_0 = 0$$

Faraday's Law implies

$$\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega} = \frac{n\hat{n}\times\vec{E}_0}{c} \qquad \hat{n} = \frac{\vec{k}}{\left|\vec{k}\right|}$$

where n is the index of refraction

Energy and Energy Flux

Poynting vector for the solution is

$$\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^* = \frac{n}{2\mu c}\left|\vec{E}_0\right|^2 \hat{n} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left|\vec{E}_0\right|^2 \hat{n}$$

Energy Density

$$u = \frac{1}{4}\left[\varepsilon \vec{E} \cdot \vec{E}^* + \frac{1}{\mu}\vec{B} \cdot \vec{B}^*\right] = \frac{\varepsilon}{2}\left|\vec{E}_0\right|^2$$

Consistency

$$\left|\vec{S}\right| = cu / n$$

Note that the units are correct

Stress Tensor for Plane Wave

Assume free-space conditions and align z-axis with $\vec{k}$

$$\varepsilon_0 \begin{pmatrix} E_x E_x + c^2 B_x B_x & E_x E_y + c^2 B_x B_y & E_x E_z + c^2 B_x B_z \\ E_y E_x + c^2 B_y B_x & E_y E_y + c^2 B_y B_y & E_y E_z + c^2 B_y B_z \\ E_z E_x + c^2 B_z B_x & E_z E_y + c^2 B_z B_y & E_z E_z + c^2 B_z B_z \end{pmatrix} - \frac{\varepsilon_0}{2} \begin{pmatrix} \vec{E} \cdot \vec{E} + \\ c^2 \vec{B} \cdot \vec{B} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{\varepsilon_0}{2} \left| \vec{E}_0 \right|^2 \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = -\frac{\varepsilon_0}{2} \left| \vec{E}_0 \right|^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Does this answer make sense?

Radiation Pressure in free space (force directed in the z-direction!)

$$\frac{\varepsilon_0}{2} \left| \vec{E}_0 \right|^2$$

Polarization Variables

Formula

$$\vec{k} \cdot \vec{E}_0 = 0 \rightarrow \vec{k} \cdot e^{i\theta} \vec{E}_0 = 0, \qquad \forall\, \theta$$

Allows for potential phase shift between transverse "vector" components of the wave. As there are two distinct directions normal to $\vec{k}$ and two potential phases, a complete description of the radiation (including the overall wave phase) involves 4 quantities, the Stokes parameters. They can be thought of as the amplitudes and phases in the two directions. They depend on the polarization basis chosen

Let the two (real) unit vectors normal to $\vec{k}$ be $\vec{e}_1$ and $\vec{e}_2$, such that

$$\vec{k} = \vec{e}_1 \times \vec{e}_2$$

Linear Polarization Basis

No phase difference between two components

$$\vec{E}\left(\vec{x},t\right)=\left[\vec{e}_1E_1+\vec{e}_2E_2\right]e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

If E_1 and E_2 purely real

$$\left|\vec{E}\right|=\sqrt{E_1^2+E_2^2}$$

E-field direction of a snapshot is constant in space. Polarization direction angle is

$$\tan^{-1}\frac{E_2}{E_1}$$

with respect to the coordinates specified by the basis vectors.

Circular Polarization Basis

Phase difference of $\pm\pi/2 = \pm i$ between two components

$$\vec{E}\left(\vec{x},t\right) = \left[\vec{e}_+ E_+ + \vec{e}_- E_-\right] e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}, \qquad \vec{e}_\pm \equiv \frac{1}{\sqrt{2}}\left(\vec{e}_1 \pm i\vec{e}_2\right)$$

If E_+ and E_- purely real

$$\left|\vec{E}\right| = \sqrt{E_+^2 + E_-^2}$$

but E-field direction of a snapshot rotates in space. Polarization is elliptical and field rotates with a handedness depending on whether

$$\left|E_1\right| < \left|E_2\right| \qquad \text{or} \qquad \left|E_1\right| > \left|E_2\right|$$

with respect to the coordinates specified by the basis vectors.

General Wave Field

$$\vec{E}\left(\vec{x},t\right)=\left[\vec{e}_1 a_1 e^{i\delta_1}+\vec{e}_2 a_2 e^{i\delta_2}\right]e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

The Stokes parameters are

$$s_0=\left|\vec{e}_1\cdot\vec{E}\right|^2+\left|\vec{e}_2\cdot\vec{E}\right|^2=a_1^2+a_2^2$$

$$s_1=\left|\vec{e}_1\cdot\vec{E}\right|^2-\left|\vec{e}_2\cdot\vec{E}\right|^2=a_1^2-a_2^2$$

$$s_2=2\,\mathrm{Re}\left[\left(\vec{e}_1\cdot\vec{E}\right)^*\left(\vec{e}_2\cdot\vec{E}\right)\right]=2a_1a_2\cos\left(\delta_2-\delta_1\right)$$

$$s_3=2\,\mathrm{Im}\left[\left(\vec{e}_1\cdot\vec{E}\right)^*\left(\vec{e}_2\cdot\vec{E}\right)\right]=2a_1a_2\cos\left(\delta_2-\delta_1\right)$$

In Circular Basis

$$\vec{E}\left(\vec{x},t\right)=\left[\vec{e}_{+}a_{+}e^{i\delta_{+}}+\vec{e}_{-}a_{-}e^{i\delta_{-}}\right]e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

The Stokes parameters are

$$s_0=\left|\vec{e}_{+}^{\,*}\cdot\vec{E}\right|^2+\left|\vec{e}_{-}^{\,*}\cdot\vec{E}\right|^2=a_{+}^2+a_{-}^2$$

$$s_1=2\,\mathrm{Re}\left[\left(\vec{e}_{+}^{\,*}\cdot\vec{E}\right)^*\left(\vec{e}_{-}^{\,*}\cdot\vec{E}\right)\right]=2a_{+}a_{-}\cos\left(\delta_{-}-\delta_{+}\right)$$

$$s_2=2\,\mathrm{Im}\left[\left(\vec{e}_{+}^{\,*}\cdot\vec{E}\right)^*\left(\vec{e}_{-}^{\,*}\cdot\vec{E}\right)\right]=2a_{+}a_{-}\sin\left(\delta_{-}-\delta_{+}\right)$$

Generally

$$s_0^2=s_1^2+s_2^2+s_3^2$$

Reflection/Refraction

At the boundary between two media of index of refraction n and n' must have continuity of field:

Incident:

$$\vec{E}\left(\vec{x},t\right)=\vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

$$\vec{B}\left(\vec{x},t\right)=\sqrt{\mu\varepsilon}\,\frac{\vec{k}\times\vec{E}}{\left|\vec{k}\right|}$$

Refracted:

$$\vec{E}'\left(\vec{x},t\right)=\vec{E}_0' e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

$$\vec{B}\left(\vec{x},t\right)=\sqrt{\mu'\varepsilon'}\,\frac{\vec{k}'\times\vec{E}'}{\left|\vec{k}'\right|}$$

Reflected:

$$\vec{E}''(\vec{x},t) = \vec{E}_0'' e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

$$\vec{B}(\vec{x},t) = \sqrt{\mu\varepsilon}\,\frac{\vec{k}''\times\vec{E}''}{\left|\vec{k}''\right|}$$

Magnitudes:

$$\left|\vec{k}\right| = \left|\vec{k}''\right| = k = \omega\sqrt{\mu\varepsilon}$$

$$\left|\vec{k}'\right| = k' = \omega\sqrt{\mu'\varepsilon'}$$

Must have:

$$\left(\vec{k}\cdot\vec{x}\right)_{z=0} = \left(\vec{k}'\cdot\vec{x}\right)_{z=0} = \left(\vec{k}''\cdot\vec{x}\right)_{z=0}$$

Snell's Law

$$k \sin i = k' \sin r = k'' \sin r'$$

And so

$$i = r'$$

$$\frac{\sin i}{\sin r} = \frac{k'}{k} = \sqrt{\frac{\mu' \varepsilon'}{\mu \varepsilon}} = \frac{n'}{n}$$