

Physics 704/804 Electromagnetic Theory II

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Retarded Solutions for Potentials



$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \phi = -\frac{\rho}{\epsilon_0}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \vec{A} = -\mu_0 \vec{J}$$

$$\phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' dt' \frac{\rho(\vec{x}', t')}{|\vec{x} - \vec{x}'|} \delta(|\vec{x} - \vec{x}'|/c - t + t')$$

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' dt' \frac{\vec{J}(\vec{x}', t')}{|\vec{x} - \vec{x}'|} \delta(|\vec{x} - \vec{x}'|/c - t + t')$$

Tip: Leave the delta function in its integral form to do derivations.
Don't have to remember complicated delta-function rules

Retarded Solutions for Fields



$$\phi(\vec{x}, t) = \frac{1}{8\pi^2 \epsilon_0} \int d^3x' dt' d\omega \frac{\rho(\vec{x}', t')}{|\vec{x} - \vec{x}'|} e^{i\omega[|\vec{x} - \vec{x}'|/c - (t - t')]} \quad (1)$$

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{8\pi^2} \int d^3x' dt' d\omega \frac{\vec{J}(\vec{x}', t')}{|\vec{x} - \vec{x}'|} e^{i\omega[|\vec{x} - \vec{x}'|/c - (t - t')]} \quad (2)$$

Evaluation can be expedited by noting and using the symmetry of the Green function and using relations such as

$$\frac{\partial}{\partial t} f(t - t') = -\frac{\partial}{\partial t'} f(t - t')$$

$$\frac{\partial}{\partial \vec{x}} f(|\vec{x} - \vec{x}'|) = -\frac{\partial}{\partial \vec{x}'} f(|\vec{x} - \vec{x}'|)$$

Yields *Jackson* 6.51-6.52

$$\begin{aligned}
 \vec{E}(\vec{x}, t) &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\
 &= \frac{1}{8\pi^2\epsilon_0} \int d^3x' dt' d\omega \frac{-\vec{\nabla}' \rho(\vec{x}', t')}{|\vec{x} - \vec{x}'|} e^{i\omega[|\vec{x} - \vec{x}'|/c - (t - t')]} \\
 &\quad - \frac{1}{8\pi^2\epsilon_0} \int d^3x' dt' d\omega \frac{\partial \vec{J}(\vec{x}', t') / \partial t'}{c^2 |\vec{x} - \vec{x}'|} e^{i\omega[|\vec{x} - \vec{x}'|/c - (t - t')]} \\
 \vec{B}(\vec{x}, t) &= \vec{\nabla} \times \vec{A} \\
 &= \frac{\mu_0}{8\pi^2} \int d^3x' dt' d\omega \frac{\vec{\nabla}' \times \vec{J}(\vec{x}', t')}{|\vec{x} - \vec{x}'|} e^{i\omega[|\vec{x} - \vec{x}'|/c - (t - t')]}
 \end{aligned}$$

Direct computation, using the unprimed variables, gives

$$\frac{\partial}{\partial \vec{x}} \frac{e^{i\omega|\vec{x}-\vec{x}'|/c}}{|\vec{x}-\vec{x}'|} = -\frac{e^{i\omega|\vec{x}-\vec{x}'|/c}}{|\vec{x}-\vec{x}'|^2} \hat{R} + \frac{(i\omega/c)e^{i\omega|\vec{x}-\vec{x}'|/c}}{|\vec{x}-\vec{x}'|} \hat{R}$$

$$\hat{R} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{|\vec{x}-\vec{x}'|}$$

Jefimenko Expressions (Jackson 6.55 and 6.56)

$$\vec{E}(\vec{x}, t) = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} = \frac{1}{8\pi^2\epsilon_0} \int d^3x' dt' d\omega \frac{\hat{R}\rho(\vec{x}', t')}{R^2} e^{i\omega[|\vec{x}-\vec{x}'|/c-(t-t')]} \\ + \frac{1}{8\pi^2\epsilon_0} \int d^3x' dt' d\omega \left[\frac{\hat{R}}{cR} \frac{\partial \rho}{\partial t'}(\vec{x}', t') - \frac{1}{c^2 R} \frac{\partial \vec{J}}{\partial t'}(\vec{x}', t') \right] e^{i\omega[|\vec{x}-\vec{x}'|/c-(t-t')]} \\$$

$$\vec{B}(\vec{x}, t) = \vec{\nabla} \times \vec{A}$$

$$\begin{aligned}
 &= \frac{\mu_0}{8\pi^2} \int d^3x' dt' d\omega \left[\vec{J}(\vec{x}', t') \times \frac{\hat{R}}{R^2} \right] e^{i\omega[|\vec{x}-\vec{x}'|/c-(t-t')]} \\
 &+ \frac{\mu_0}{8\pi^2} \int d^3x' dt' d\omega \left[\frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} \times \frac{\hat{R}}{cR} \right] e^{i\omega[|\vec{x}-\vec{x}'|/c-(t-t')]}
 \end{aligned}$$

For time independent distributions reproduce Jackson 1.5 and 5.14 (Coulomb's Law or the Biot-Savart Law for magnetic field).

Heavyside-Feynman fields for a point particle are found in the homework problem.

Energy/Power considerations



Units of electric field , electric displacement, and current density

$$[\vec{E}] = \frac{\text{Nt}}{\text{C}} \quad [\vec{J}] = \frac{\text{C}}{\text{sec} \cdot \text{m}^2} \quad [\vec{D}] = \frac{\text{C}}{\text{m}^2}$$

$$[\vec{E}][\vec{J}] = \frac{J}{\text{sec} \cdot \text{m}^3} \quad [\vec{E}][\vec{D}] = \frac{J}{\text{m}^3}$$

Power delivered from electromagnetic field to a current element per unit volume involves E and J forms (magnetic fields cannot do work on a current because the force is perpendicular to the current). Power per unit volume must be

$$\begin{aligned} (E_x dx + E_y dy + E_z dz) \wedge (J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy) \\ = (\vec{E} \cdot \vec{J}) dx \wedge dy \wedge dz \end{aligned}$$

Energy Conservation



Energy/volume involves (exterior) multiplication of E and D . The exact formula comes from the Maxwell Equations:

$$\begin{aligned} \int_V \omega_E^1 \wedge \omega_J^2 &= \int_V \omega_E^1 \wedge \left(d\omega_H^1 - \frac{\partial}{\partial t} \omega_D^2 \right) - \int_V \left(d\omega_E^1 + \frac{\partial}{\partial t} \omega_B^2 \right) \wedge \omega_H^1 \\ &= - \int_V d \left(\omega_E^1 \wedge \omega_H^1 \right) - \int_V \left[\omega_E^1 \wedge \frac{\partial}{\partial t} \omega_D^2 + \omega_H^1 \wedge \frac{\partial}{\partial t} \omega_B^2 \right] \end{aligned}$$

The RHS terms define the [1] Poynting (Energy Flux) form

$$\omega_E^1 \wedge \omega_H^1 = \omega_{\vec{E} \times \vec{H}}^2$$

For linear materials have [2] energy density form

$$\frac{1}{2} \omega_E^1 \wedge \omega_D^2 + \frac{1}{2} \omega_H^1 \wedge \omega_B^2 = \left(\frac{\vec{E} \cdot \vec{D}}{2} + \frac{\vec{H} \cdot \vec{B}}{2} \right) dx \wedge dy \wedge dz$$

In Vector Form



Energy Density (J/m^3)

$$u = \frac{\vec{E} \cdot \vec{D}}{2} + \frac{\vec{H} \cdot \vec{B}}{2}$$

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}$$

Poynting Vector (J/sec m^2)

$$\vec{S} = \vec{E} \times \vec{H}$$