Physics 704/804 Electromagnetic Theory II

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Retarded Solutions for Fields

$$\phi(\vec{x},t) = \frac{1}{8\pi^2\varepsilon_0}\int d^3x' dt' d\omega \frac{\rho(\vec{x}',t')}{|\vec{x}-\vec{x}'|} e^{i\omega\left[|\vec{x}-\vec{x}'|/c-(t-t')\right]}$$

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{8\pi^2}\int d^3x' dt' d\omega \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|} e^{i\omega\left[|\vec{x}-\vec{x}'|/c-(t-t')\right]}$$

Evaluation can be expedited by noting and using the symmetry of the Green function and using relations such as

$$\frac{\partial}{\partial t} f(t-t') = -\frac{\partial}{\partial t'} f(t-t')$$

$$\frac{\partial}{\partial \vec{x}} f(|\vec{x}-\vec{x}'|) = -\frac{\partial}{\partial \vec{x}'} f(|\vec{x}-\vec{x}'|)$$

Direct computation, using the unprimed variables, gives

$$\frac{\partial}{\partial \vec{x}} \frac{e^{i\omega|\vec{x}-\vec{x}'|/c}}{|\vec{x}-\vec{x}'|} = -\frac{e^{i\omega|\vec{x}-\vec{x}'|/c}}{|\vec{x}-\vec{x}'|^2}\hat{R} + \frac{(i\omega/c)e^{i\omega|\vec{x}-\vec{x}'|/c}}{|\vec{x}-\vec{x}'|}\hat{R}$$

$$\hat{R} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{|\vec{x}-\vec{x}'|}$$

Jefimenko Expressions (Jackson 6.55 and 6.56)

$$\vec{E}(\vec{x},t) = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} = \frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega \frac{\hat{R}\rho(\vec{x}',t')}{R^2} e^{i\omega[|\vec{x}-\vec{x}'|/c-(t-t')]}$$

$$+\frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega \left[\frac{\hat{R}}{cR}\frac{\partial\rho}{\partial t'}(\vec{x}',t') - \frac{1}{c^2R}\frac{\partial\vec{J}}{\partial t'}(\vec{x}',t')\right] e^{i\omega[|\vec{x}-\vec{x}'|/c-(t-t')]}$$

$$\vec{B}(\vec{x},t) = \vec{\nabla} \times \vec{A}$$

$$= \frac{\mu_0}{8\pi^2} \int d^3x' dt' d\omega \left[\vec{J}(\vec{x}',t') \times \frac{\hat{R}}{R^2} \right] e^{i\omega\left[|\vec{x}-\vec{x}'|/c-(t-t')\right]}$$

$$+ \frac{\mu_0}{8\pi^2} \int d^3x' dt' d\omega \left[\frac{\partial \vec{J}(\vec{x}',t')}{\partial t'} \times \frac{\hat{R}}{cR} \right] e^{i\omega\left[|\vec{x}-\vec{x}'|/c-(t-t')\right]}$$

For time independent distributions reproduce Jackson 1.5 and 5.14 (Coulomb's Law or the Biot-Savart Law for magnetic field).

Heavyside-Feynman fields for a point particle are found in the homework problem.

Energy/Power considerations

Units of electric field , electric displacement, and current density

$$\left[\vec{E}\right] = \frac{\mathrm{Nt}}{C} \qquad \left[\vec{J}\right] = \frac{C}{\mathrm{sec}\cdot \mathrm{m}^2} \qquad \left[\vec{D}\right] = \frac{C}{\mathrm{m}^2}$$

$$\left[\vec{E}\right]\left[\vec{J}\right] = \frac{J}{\mathrm{sec}\cdot \mathrm{m}^3} \qquad \left[\vec{E}\right]\left[\vec{D}\right] = \frac{J}{\mathrm{m}^3}$$

Power delivered from electromagnetic field to a current element per unit volume involves *E* and *J* forms (magnetic fields cannot do work on a current because the force is perpendicular to the current). Power per unit volume must be

$$\left(E_x dx + E_y dy + E_z dz\right) \wedge \left(J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy\right)$$

$$= \left(\vec{E} \cdot \vec{J}\right) dx \wedge dy \wedge dz$$

Energy Conservation

Energy/volume involves (exterior) multiplication of E and D. The exact formula comes from the Maxwell Equations:

$$\int_V \omega_E^1 \wedge \omega_J^2 = \int_V \omega_E^1 \wedge \left(d\,\omega_H^1 - \frac{\partial}{\partial t}\omega_D^2 \right) - \int_V \left(d\,\omega_E^1 + \frac{\partial}{\partial t}\omega_B^2 \right) \wedge \omega_H^1$$

$$= -\int_V d\left(\omega_E^1 \wedge \omega_H^1 \right) - \int_V \left[\omega_E^1 \wedge \frac{\partial}{\partial t}\omega_D^2 + \omega_H^1 \wedge \frac{\partial}{\partial t}\omega_B^2 \right]$$

The RHS terms define the [1] Poynting (Energy Flux) form

$$\omega_E^1 \wedge \omega_H^1 = \omega_{\vec{E}\times\vec{H}}^2$$

For linear materials have [2] energy density form

$$\frac{1}{2}\omega_E^1 \wedge \omega_D^2 + \frac{1}{2}\omega_H^1 \wedge \omega_B^2 = \left(\frac{\vec{E}\cdot\vec{D}}{2} + \frac{\vec{H}\cdot\vec{B}}{2} \right) dx \wedge dy \wedge dz$$

In Vector Form

Energy Density (J/m^3)

$$u = \frac{\vec{E} \cdot \vec{D}}{2} + \frac{\vec{H} \cdot \vec{B}}{2}$$

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}$$

Poynting Vector (J/sec m^2)

$$\vec{S} = \vec{E} \times \vec{H}$$

Momentum Conservation

Momentum is a vector quantity. To describe momentum densities need (what could be more natural) vector valued forms. Mechanical momentum delivered to a load by an electromagnetic field is

$$\frac{d\vec{p}_{mech}}{dt} = \int_V \left[\rho \vec{E} + \vec{J} \times \vec{B} \right] dxdydx$$

Define the vector 3-form force density to be

$$\left[\rho \vec{E} + \vec{J} \times \vec{B} \right] dx \wedge dy \wedge dz$$

Momentum Flux density must have nine components

$$\hat{e}_i T_{ij} \varepsilon_{jkl} dx^k \otimes dx^l = \hat{e}_i \left[T_{i1} dy \wedge dz + T_{i2} dz \wedge dx + T_{i3} dx \wedge dy \right]$$