

Physics 704/804 Electromagnetic Theory II

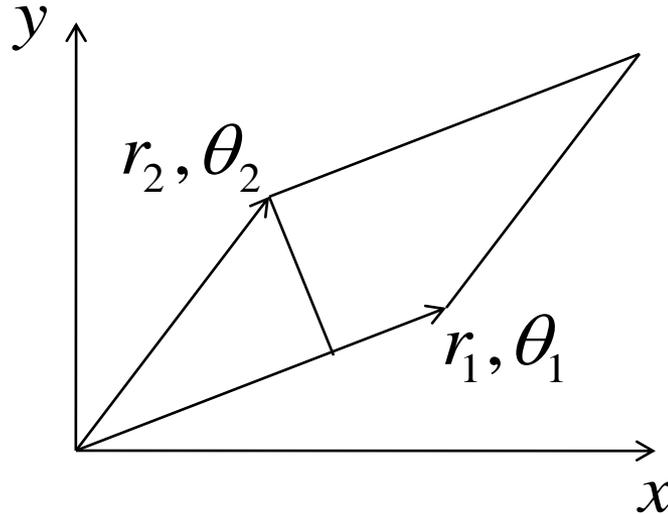
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Projected Area Argument

Example: projection into x - y plane



$$\begin{aligned} A &= r_1 r_2 \sin(\theta_2 - \theta_1) \\ &= r_1 r_2 (\sin \theta_2 \cos \theta_1 - \cos \theta_2 \sin \theta_1) \\ &= x_1 y_2 - x_2 y_1 \end{aligned}$$

Faraday's Law



Magnetic Flux Density (Magnetic Induction) Form

$$\omega_B^2 \equiv \varepsilon_{ijk} B^i dx^j \otimes dx^k = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

Magnetic Flux through a closed loop L ; for any S with $L = \partial S$
(note, independent of the actual surface chosen, why?)

$$\phi_B = \int_S \omega_B^2$$

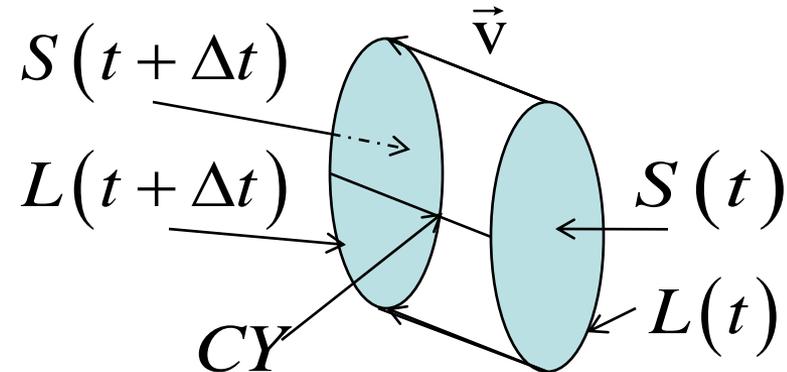
Faraday's Law: induced emf, \mathbf{E} is

$$\int_L \omega_{E'}^1 \equiv \int_L E'_x dx + E'_y dy + E'_z dz = \mathbf{E} = -k \frac{d\phi_B}{dt}$$

The ' on the electric field indicates must use value *at rest with loop* when performing the integral

Analysis of Moving Loop

Assume loop moves with a constant velocity \vec{v}



and let the equation for the loop at time t be given as a function of some parameter s , *e.g.* the path length.

$$\vec{x}(s, t) = \vec{x}_0(s) + \vec{v}t$$

is a perfectly good two-variable parameterization of the surface CY joining the loop at the two different times.

$$\begin{aligned}
 \frac{d\phi_B}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\phi_B(t + \Delta t) - \phi_B(t)}{\Delta t} \\
 &= \lim_{\Delta t \rightarrow 0} \int_{s(t+\Delta t)} \frac{B_z(t + \Delta t, \vec{x}) - B_z(t, \vec{x})}{\Delta t} dx \wedge dy + \text{cyclic perms} \\
 &\quad - \lim_{\Delta t \rightarrow 0} \int_{CY} \frac{B_z(t, \vec{x})}{\Delta t} dx \wedge dy + \text{cyclic perms}
 \end{aligned}$$

Where CY is the cylindrical surface joining the loops at the different times. Now on CY

$$dx = \frac{dx_0}{ds} ds + v_x dt \quad dx \wedge dy = \left(\frac{dx_0}{ds} v_y - \frac{dy_0}{ds} v_x \right) ds \wedge dt$$

$$dy = \frac{dy_0}{ds} ds + v_y dt \quad \therefore dy \wedge dz = \left(\frac{dy_0}{ds} v_z - \frac{dz_0}{ds} v_y \right) ds \wedge dt$$

$$dz = \frac{dz_0}{ds} ds + v_z dt \quad dz \wedge dx = \left(\frac{dz_0}{ds} v_x - \frac{dx_0}{ds} v_z \right) ds \wedge dt$$

For small Δt , little error is made by replacing the first term by the partial time derivative, and by replacing the values of the magnetic field on the surface by the values at time t , and integrating the time integral trivially. In the limit the differences vanish; an exact (Jackson 5.137) expression results.

$$\begin{aligned}
 \frac{d\phi_B}{dt} &= \int_{s(t)} \frac{\partial B_z(t, \vec{x})}{\partial t} dx \wedge dy + \text{cyclic perms} \\
 &\quad - \int_{L(t)} \left(\frac{dx_0}{ds} v_y - \frac{dy_0}{ds} v_x \right) B_z(t, \vec{x}) ds + \text{cyclic perms} \\
 &= \int_{s(t)} \frac{\partial B_z(t, \vec{x})}{\partial t} dx \wedge dy + \text{cyclic perms} \\
 &\quad - \int_{L(t)} \left(\vec{v} \times \vec{B}(t, \vec{x}) \right)_x dx + \text{cyclic perms}
 \end{aligned}$$

Galilean Invariance



Usual formulation (pre relativity): Laws of physics (e.g., the force vector) must be invariant to Galilean transformations

$$\vec{x}' = \vec{x} - \vec{v}t$$

$$t' = t$$

where the prime frame moves with respect to the unprime frame with the velocity \vec{v} . The inverse transformation is

$$\vec{x} = \vec{x}' + \vec{v}t'$$

$$t = t'$$

In the rest frame of the loop Faraday's law is

$$\mathbf{E}' = \int_{S'} d\omega_{E'}^1 = -k \frac{d\phi_{B'}}{dt'} = -k \int_{S'} \frac{\partial \omega_{B'}^2}{\partial t'}$$

Transformation of magnetic field

$$\vec{B}'(t', \vec{x}') \approx \vec{B}(t' + \vec{v} \cdot \vec{x}' / c^2, \vec{x}' + \vec{v}t') \approx \vec{B}(t', \vec{x}' + \vec{v}t')$$

where the unprimed magnetic field function describes the magnetic field in the *laboratory frame*.

$$\begin{aligned} \mathbf{E}' &= \int_{S'} d\omega_{E'}^1 = -k \int_{S'} \left[\frac{\partial \omega_B^2(t', \vec{x}' + \vec{v}t')}{\partial t} + \vec{v} \cdot \frac{\partial \omega_B^2(t', \vec{x}' + \vec{v}t')}{\partial \vec{x}} \right] \\ &= \int_{S'} d\omega_E^1(t', \vec{x}' + \vec{v}t') + k \int_{S'} d\omega_{\vec{v} \times \vec{B}}^1(t', \vec{x}' + \vec{v}t') \end{aligned}$$

Therefore,

$$\vec{E}' = \vec{E} + k(\vec{v} \times \vec{B})$$

Comparing to the Lorentz force law, k must be 1!

Generalized “Faraday Law”



In the lab frame, the following expression for the electromotive force applies. It may be considered to be a Galilean invariant, or if you prefer, invariant for Galilean transformations in the limit of velocities small compared to light. The expression encompasses both “motional emf” and “regular emf”. \vec{v} gives the instantaneous velocity of the loop integration element. Strictly speaking, we have only verified when \vec{v} is a constant velocity, but I believe it applies when the velocity is time dependent (Extra credit, give me a general proof!)

$$\begin{aligned} \mathcal{E} &= \oint_{L(t)} \omega_{\vec{E} + \vec{v} \times \vec{B}}^1 \equiv \oint_{L(t)} \left(\vec{E} + \vec{v} \times \vec{B} \right)_x dx + \left(\vec{E} + \vec{v} \times \vec{B} \right)_y dy + \left(\vec{E} + \vec{v} \times \vec{B} \right)_z dz \\ &= -\frac{d\phi_B}{dt} = -\frac{d}{dt} \int_{S(t)} \omega_B^2 \end{aligned}$$

Differential Form of Faraday's Law



For any loop and surface fixed in the frame the fields are defined in has

$$\oint_L \omega_E^1 = - \int_S \frac{\partial B_z(t, \vec{x})}{\partial t} dx \wedge dy + \text{cyclic perms} = - \int_S \frac{\partial \omega_B^2}{\partial t}$$

Stokes Theorem gives

$$\oint_L \omega_E^1 = \int_S d\omega_E^1 \rightarrow \int_S d\omega_E^1 + \frac{\partial \omega_B^2}{\partial t} = 0$$

This equation is true for all possible choices of L and S , which means the integrand must be 0. In form language and conventional vector language

$$d\omega_E^1 = - \frac{\partial}{\partial t} \omega_B^2 \qquad \vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$