Physics 704/804
Mid Term Examination
Due Mar 17, 2009

1) Let $f(z)$ be a complex function of a single complex variable z. In complex function theory, $f(z)$ is called *analytic* at a point *z* if the complex derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

exists. In particular, this limiting ratio must be independent of the direction in the complex plane from which $\Delta z \to 0$.

a) If such a function is resolved into its real and imaginary parts $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where x and y are the real and imaginary parts of the complex variable z, show that in order for the derivative to exist, the Cauchy-Riemann conditions $\partial u / \partial x = \partial v / \partial y$ and $\partial u / \partial y = -\partial v / \partial x$ must apply.

b) Given an analytic $f(z)$, suppose the $\mathbf{R}^2$ forms $\omega^1_{\mathrm{Re}}(x,y) = u(x,y)\,dx - v(x,y)\,dy$ and $\omega^1_{\mathrm{Im}}(x,y) = u(x,y)\,dy + v(x,y)\,dx$ are defined. What are $d\omega^1_{\mathrm{Re}}$ and $d\omega^1_{\mathrm{Im}}$?

c) Write the usual complex integral along a path *P*,

$$\int_P f(z)dz,$$

in terms of ω^1_{Re} and ω^1_{Im}.

d) For $f(z)$ analytic, and *C* a closed curve in the (complex) plane, prove Cauchy's Theorem $\oint_C f(z)dz = 0$. (Hint: Generalized Stokes Theorem)

2) Four plane electromagnetic waves in free space are represented by the following expressions

$$\vec{E}_1 = \mathrm{Re}\left[E_1 \hat{x} e^{i(kz-\omega t)}\right] \qquad \vec{B}_1 = \mathrm{Re}\left[(E_1/c)\,\hat{y} e^{i(kz-\omega t)}\right]$$

$$\vec{E}_2 = \mathrm{Re}\left[E_2 \hat{y} e^{i(kz-\omega t+\alpha)}\right] \qquad \vec{B}_2 = -\mathrm{Re}\left[(E_2/c)\,\hat{x} e^{i(kz-\omega t+\alpha)}\right]$$

$$\vec{E}_3 = \mathrm{Re}\left[E_3 \hat{x} e^{i(kz-\omega t+\alpha)}\right] \qquad \vec{B}_3 = \mathrm{Re}\left[(E_3/c)\,\hat{y} e^{i(kz-\omega t+\alpha)}\right]$$

$$\vec{E}_4 = \mathrm{Re}\left[E_1 \hat{x} e^{i(-kz-\omega t)}\right] \qquad \vec{B}_4 = -\mathrm{Re}\left[(E_1/c)\,\hat{y} e^{i(-kz-\omega t)}\right]$$

a) Compute the instantaneous Poynting vector $\vec{S}$ and the time-averaged Poynting vector $\langle \vec{S} \rangle$ for the superposition of waves one and two. Show that these quantities are just the sums of the corresponding quantities for the waves taken separately. Why?

b) Sketch the ellipses traced out by the electric field and magnetic field vectors for a superposition of waves one and two in the specific case $E_2 = 2E_1$ and $\alpha = 30^\circ$.

c) Compute $\vec{S}$ and $\langle \vec{S} \rangle$ for the superposition of waves one and three. Compare with the results in a) and explain the difference.

d) Compute $\vec{S}$ for the superposition of waves one and four. Show that the energy oscillates, being purely electric then purely magnetic. Show that $\langle \vec{S} \rangle$ vanishes. Explain. (20 Points)

3) Jackson, Problem 7.19 (20 Points)

4) a) Compute the mode cutoff frequencies for TE and TM modes propagating through a waveguide with the "half-circle" transverse geometry of Figure 1.

b) For the lowest frequency propagating mode, explicitly display ψ, and the transverse electric and transverse magnetic fields in terms of the relevant special functions. Express your results using the notations in Jackson, Section 8.7.

c) What is the wave impedance for the lowest mode?

d) Find the lowest cutoff frequency for the guide shaped as a "quarter-circle" as in Figure 2. (40 Points.)

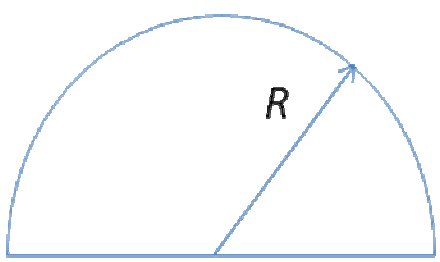

Figure 1

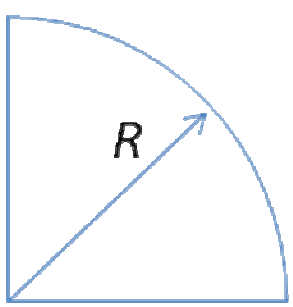

Figure 2