Physics 704/804
Final Examination
May 05, 2009

1) a) Let S be a *closed* surface $(\partial S = \varnothing)$. Using the appropriate Maxwell equation, show that the flux integral

$$\phi_{\vec{B}} = \int_S \omega_{\vec{B}}^2$$

vanishes. (10 pts)

b) Numerically compute the intensity (W/m^2) of a linearly polarized plane wave with a peak electric field of 0.1 V/m. Numerically compute the electric field of an observer 3 m from a 50 W point source (like a light bulb) of electromagnetic radiation. For the purposes of this calculation assume all the emission happens at a single frequency. What is the electric field at 6 m away? (10 pts)

c) Compute numerically the lowest (*fundamental*) oscillation frequency of a cavity whose transverse geometry is as in Figure 1 and whose longitudinal length is L. Take $R = 10$ cm and $L = 30$ cm for your dimensions and assume the cavity encloses free space. Utilize any result from the posted mid-term that you wish in completing this problem. (10 pts)

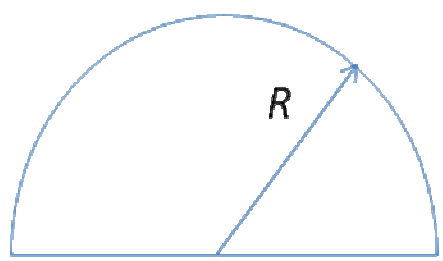

Figure 1

2) Suppose a charge q moves on a circle at non-relativistic velocities

$$\vec{x}(t) = a(\cos\omega t\hat{x} + \sin\omega t\hat{y}),$$

and that the emitted radiation is well described by long wavelength dipole emission.

a) What are the dipole moments describing this emission? (8 pts)
b) What is the far field radiation pattern,

$$\frac{dP}{d\Omega}(\theta,\phi)$$

where θ and ϕ are the standard spherical coordinates? (8 pts)

c) Specialize to the three observation directions $\hat{x}$ with $\theta = \pi/2$ and $\phi = 0$, $\hat{y}$ with $\theta = \pi/2$ and $\phi = \pi/2$, and $\hat{z}$ with $\theta = 0$. Define a suitable set of linear polarization vectors for each observation direction and label them with subscripts, e.g. $\varepsilon_{1\hat{x}} = ??$ and $\varepsilon_{2\hat{x}} = ??$, etc., so that each appropriate combination of the three vectors (for example the set $\varepsilon_{1\hat{x}}, \varepsilon_{2\hat{x},} \hat{x}$) form a right-handed set. (8 pts)

d) Complete the table by computing the power per unit solid angle into each polarization at each observation direction, normalized by the *total* power per unit solid angle in the $\hat{z}$ direction ($\theta = 0$). You may find the footnote at the bottom of page 411 of Jackson helpful. Two straightforward results are already entered. Explain your results. (8 pts)

Observation Direction	ε_1	ε_2	$\vec{\varepsilon}_+ = \frac{\vec{\varepsilon}_1 + i\vec{\varepsilon}_2}{\sqrt{2}}$	$\vec{\varepsilon}_- = \frac{\vec{\varepsilon}_1 - i\vec{\varepsilon}_2}{\sqrt{2}}$
$\hat{z}$			1	0
$\hat{x}$				
$\hat{y}$				

Table I: Power per unit solid angle into each polarization, normalized by the total power per unit solid angle in the $\hat{z}$ direction.

e) What is the total power radiated? Is your result consistent with Larmor's theorem? (8 pts)

f) Extra credit: What is the polarization of the emission in the $-\hat{z}$ direction? What happens to the table if the direction of motion is reversed? (5 pts)

3) a) Show explicitly the invariance of the wave operator

$$\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

for two coordinate systems related by the Lorentz transformation

$$\begin{aligned} ct' &= \gamma\left(ct - \beta x\right) \\ x' &= \gamma\left(x - \beta ct\right) \\ y' &= y \\ z' &= z. \end{aligned}$$

(10 pts)

b) Show the Lorentz boost in part a) is generated by the matrix

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

as $L_x(\xi) = e^{-\xi K_1}$ where $\xi = \tanh^{-1}\beta$ is the rapidity, by showing

$$e^{-\xi K_1} = \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ (10 pts)

c) Show rapidities, for combined transformations in the same direction, add algebraically by performing the matrix multiplication $L_x(\xi_1 + \xi_2) = L_x(\xi_1) L_x(\xi_2)$. (10 pts)