Physics 604 Electromagnetic Theory I

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Surface Density

$$\sigma_M = \vec{n} \cdot \vec{M}$$

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' + \frac{1}{4\pi}\oint \frac{\vec{n}' \cdot \vec{M}(\vec{x}')\, da'}{|\vec{x}-\vec{x}'|}$$

Vector Potential Solution

$$\vec{\nabla} \times \vec{H} = \vec{\nabla} \times \left(\vec{B} / \mu_0 - M\right) = 0$$

$$\nabla^2 \vec{A} = -\mu_0 J_M \qquad \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$$

with surface term

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' + \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{x}') \times \vec{n}'}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$$

Uniformly Magnetized Sphere

$$\vec{M} = M_0 \hat{z} \rightarrow \sigma_M = M_0 \cos\theta$$

$$\Phi_M\left(\vec{x}\right) = -\frac{1}{4\pi}\int \frac{\vec{\nabla}' \cdot \vec{M}\left(\vec{x}'\right)}{\left|\vec{x} - \vec{x}'\right|} d^3\vec{x}' + \frac{1}{4\pi}\oint \frac{d\Omega' \cos\theta}{\left|\vec{x} - \vec{x}'\right|}$$

$$\Phi_M\left(r, \theta\right) = 0 + \frac{M_0 a^2}{3}\frac{r_<}{r_>^2}\cos\theta$$

inside sphere $\vec{H}_{in} = -\frac{1}{3}$ $\vec{B}_{in} = \frac{2\mu_0}{3}\vec{M}$

outside sphere $\Phi_M = \frac{M_0 a^3}{3}\frac{\cos\theta}{r^2}$

$$\vec{m} = \frac{4\pi a^3}{3}\vec{M}$$

Alternate Calculations

$$\Phi_M(r,\theta) = -\frac{1}{4\pi}\vec{\nabla}\cdot\int\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}' = -\frac{M_0}{4\pi}\frac{\partial}{\partial z}\int_0^\infty r'^2 dr'\int\frac{d\Omega'}{|\vec{x}-\vec{x}'|}$$

$$= -M_0\cos\theta\frac{\partial}{\partial r}\int_0^a\frac{r'^2}{r_>}dr'$$

with vector potential

$$\vec{M}\times\vec{n}' = M_0\sin\theta'\hat{\phi} = M_0\sin\theta'\left(-\sin\phi'\hat{x}+\cos\phi'\hat{y}\right)$$

$$A_\phi(\vec{x}) = \frac{\mu_0}{4\pi}M_0 a^2\int d\Omega'\frac{\sin\theta'\cos\phi'}{|\vec{x}-\vec{x}'|}$$

$$\sin\theta'\cos\phi' = -\sqrt{\frac{8\pi}{3}}\mathrm{Re}\left[Y_{11}(\theta',\phi')\right]$$

$$A_\phi = \frac{\mu_0}{3}M_0 a^2\left(\frac{r_<}{r_>^2}\right)\sin\theta$$

Sphere in External Field

$$\vec{B}_{in} = \vec{B}_0 + \frac{2\mu_0}{3}\vec{M}$$

$$\vec{H}_{in} = \frac{1}{\mu_0}\vec{B}_0 - \frac{1}{3}\vec{M}$$

$$\vec{B}_{in} = \mu \vec{H}_{in}$$

$$\therefore\ \vec{B}_0 + \frac{2\mu_0}{3} = \mu\left(\frac{1}{\mu_0}\vec{B}_0 - \frac{1}{3}\vec{M}\right)$$

$$\vec{M} = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu - 2\mu_0}\right)$$

For ferromagnet use Hysteresis curve plus

$$\vec{B}_{in} + 2\mu_0 \vec{H}_{in} = 3\vec{B}_0$$

Magnetic Shielding

$$\vec{H} = -\vec{\nabla}\Phi_M$$

$$\Phi_M = -H_0 r\cos\theta + \sum_{l=0}^{\infty}\frac{\alpha_l}{r^{l+1}}P_l\left(\cos\theta\right) \qquad r > b$$

$$\sum_{l=0}^{\infty}\left[\beta_l r^l + \frac{\gamma_l}{r^{l+1}}\right]P_l\left(\cos\theta\right) \qquad a < r < b$$

$$\sum_{l=0}^{\infty}\frac{\delta_l}{r^{l+1}}P_l\left(\cos\theta\right) \qquad r < a$$

$$\frac{\partial\Phi_M}{\partial\theta}\left(b+\right) = \frac{\partial\Phi_M}{\partial\theta}\left(b-\right) \qquad \frac{\partial\Phi_M}{\partial\theta}\left(a+\right) = \frac{\partial\Phi_M}{\partial\theta}\left(a-\right)$$

$$\mu_0\frac{\partial\Phi_M}{\partial r}\left(b+\right) = \mu\frac{\partial\Phi_M}{\partial\theta}\left(b-\right) \qquad \mu\frac{\partial\Phi_M}{\partial r}\left(a+\right) = \mu\frac{\partial\Phi_M}{\partial\theta}\left(a-\right)$$

$$\alpha_1 - b^3\beta_1 - \gamma_1 = b^3 H_0$$

$$\alpha_1 - \mu' b^3\beta_1 - 2\mu'\gamma_1 = -b^3 H_0$$

$$a^3\beta_1 + \gamma_1 - a^3\delta_1 = 0$$

$$\mu' a^3\beta_1 + 2\mu'\gamma_1 - a^3\delta_1 = 0$$

$$\mu' = \mu / \mu_0$$

$$\alpha_1 = \left[\frac{(2\mu'+1)(\mu'-1)}{(2\mu'+1)(\mu'+2) - 2\frac{a^3}{b^3}(\mu'-2)^2}\right]\left(b^3 - a^3\right) H_0 \to b^3 H_0$$

$$\delta_1 = \left[\frac{9\mu'}{(2\mu'+1)(\mu'+2) - 2\frac{a^3}{b^3}(\mu'-2)^2}\right]\left(b^3 - a^3\right) H_0 \to -\frac{9\mu_0}{2\mu\left(1 - \frac{a^3}{b^3}\right)} H_0$$

Faraday's Law

Magnetic Flux Density (Magnetic Induction)

Magnetic Flux through a closed loop L; for any S with $L = \partial S$ (note, independent of the actual surface chosen, why?)

$$\phi_B = \int_S \vec{B} \cdot \vec{n}\, da$$

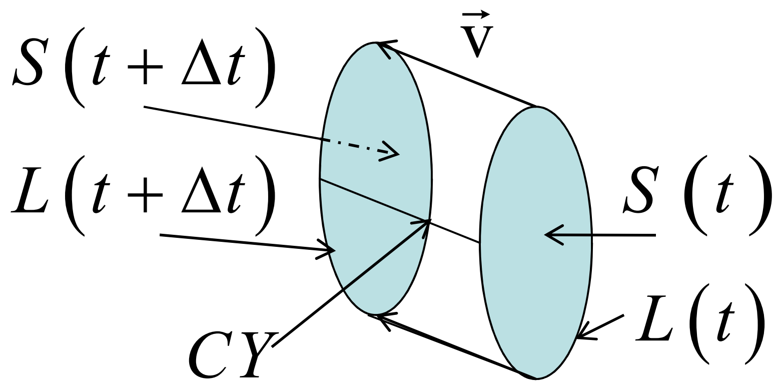
Faraday's Law: induced emf, **E** is

$$\int_{L=\partial S} E'_x dx + E'_y dy + E'_z dz = \mathsf{E} = -k \frac{d\phi_B}{dt}$$

The ' on the electric field indicates must use value *at rest with loop* when performing the integral

Analysis of Moving Loop

Assume loop moves with a constant velocity $\vec{\mathrm{v}}$



and let the equation for the loop at time *t* be given as a function of some parameter *s, e.g.* the path length.

$$\vec{x}(s,t) = \vec{x}_0(s) + \vec{\mathrm{v}}t$$

is a perfectly good two-variable parameterization of the surface *CY* joining the loop at the two different times.

$$\frac{d\phi_B}{dt} = \lim_{\Delta t \to 0} \frac{\phi_B(t+\Delta t) - \phi_B(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \int_{s(t+\Delta t)} \frac{B_z(t+\Delta t, \vec{x}) - B_z(t, \vec{x})}{\Delta t} dx \wedge dy + cyclic\ perms$$

$$- \lim_{\Delta t \to 0} \int_{CY} \frac{B_z(t, \vec{x})}{\Delta t} dx \wedge dy + cyclic\ perms$$

Where CY is the cylindrical surface joining the loops at the different times. Now on CY

$$dx = \frac{dx_0}{ds} ds + \mathrm{v}_x dt \qquad\qquad dx \wedge dy = \left(\frac{dx_0}{ds} \mathrm{v}_y - \frac{dy_0}{ds} \mathrm{v}_x \right) ds \wedge dt$$

$$dy = \frac{dy_0}{ds} ds + \mathrm{v}_y dt \qquad \therefore \qquad dy \wedge dz = \left(\frac{dy_0}{ds} \mathrm{v}_z - \frac{dz_0}{ds} \mathrm{v}_y \right) ds \wedge dt$$

$$dz = \frac{dz_0}{ds} ds + \mathrm{v}_z dt \qquad\qquad dz \wedge dx = \left(\frac{dz_0}{ds} \mathrm{v}_x - \frac{dx_0}{ds} \mathrm{v}_z \right) ds \wedge dt$$

For small Δt, little error is made by replacing the first term by the partial time derivative, and by replacing the values of the magnetic field on the surface by the values at time t, and integrating the time integral trivially. In the limit the differences vanish; an exact (Jackson 5.137) expression results.

$$\frac{d\phi_B}{dt} = \int_{s(t)} \frac{\partial B_z(t,\vec{x})}{\partial t} dx \wedge dy + cyclic\ perms$$

$$-\int_{L(t)} \left(\frac{dx_0}{ds} \mathrm{v}_y - \frac{dy_0}{ds} \mathrm{v}_x \right) B_z(t,\vec{x})\, ds + cyclic\ perms$$

$$= \int_{s(t)} \frac{\partial B_z(t,\vec{x})}{\partial t} dx \wedge dy + cyclic\ perms$$

$$-\int_{L(t)} \left(\vec{\mathrm{v}} \times \vec{B}(t,\vec{x}) \right)_x dx + cyclic\ perms$$

Galilean Invariance

Usual formulation (pre relativity): Laws of physics (e.g., the force vector) must be invariant to Galilean transformations

$$\vec{x}' = \vec{x} - \vec{\mathrm{v}}t$$
$$t' = t$$

where the prime frame moves with respect to the unprime frame with the velocity $\vec{\mathrm{v}}$. The inverse transformation is

$$\vec{x} = \vec{x}' + \vec{\mathrm{v}}t'$$
$$t = t'$$

In the rest frame of the loop Faraday's law is

$$\mathsf{E}' = -k\frac{d\phi_{B'}}{dt'} = -k\int_{S'}\frac{\partial\omega_{B'}^2}{\partial t'}$$

Transformation of magnetic field

$$\vec{B}'\left(t',\vec{x}'\right) \approx \vec{B}\left(t' + \vec{\mathrm{v}}\cdot\vec{x}'/c^2, \vec{x}' + \vec{\mathrm{v}}t'\right) \approx \vec{B}\left(t', \vec{x}' + \vec{\mathrm{v}}t'\right)$$

where the unprimed magnetic field function describes the magnetic field in the *laboratory frame*.

$$\mathsf{E}' = \int_{S'} d\,\omega^1_{E'} = -k\int_{S'} \left[\frac{\partial \omega^2_B\left(t', \vec{x}' + \vec{\mathrm{v}}t'\right)}{\partial t} + \vec{\mathrm{v}} \cdot \frac{\partial \omega^2_B\left(t', \vec{x}' + \vec{\mathrm{v}}t'\right)}{\partial \vec{x}} \right]$$

$$= \int_{S'} d\,\omega^1_E\left(t', \vec{x}' + \vec{\mathrm{v}}t'\right) + k\int_{S'} d\,\omega^1_{\mathrm{v}\times B}\left(t', \vec{x}' + \vec{\mathrm{v}}t'\right)$$

Therefore,

$$\vec{E}' = \vec{E} + k\left(\vec{\mathrm{v}} \times \vec{B}\right)$$

Comparing to the Lorentz force law, *k* must be 1!

Generalized "Faraday Law"

In the lab frame, the following expression for the electromotive force applies. It may be considered to be a Galilean invariant, or if you prefer, invariant for Galilean transformations in the limit of velocities small compared to light. The expression encompasses both "motional emf" and "regular emf". $\vec{\mathrm{v}}$ gives the instantaneous velocity of the loop integration element. Strictly speaking, we have only verified when $\vec{\mathrm{v}}$ is a constant velocity, but I believe it applies when the velocity is time dependent (Extra credit, give me a general proof!)

$$\mathsf{E} = \oint_{L(t)} \omega^1_{E+\mathrm{v}\times B} \equiv \oint_{L(t)} \left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)_x dx + \left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)_y dy + \left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)_z dz$$

$$= -\frac{d\phi_B}{dt} = -\frac{d}{dt} \int_{S(t)} \omega^2_B$$

Differential Form of Faraday's Law

For any loop and surface fixed in the frame the fields are defined in has

$$\oint_L \omega_E^1 = -\int_S \frac{\partial B_z(t,\vec{x})}{\partial t} dx \wedge dy + cyclic\ perms = -\int_S \frac{\partial \omega_B^2}{\partial t}$$

Stokes Theorem gives

$$\oint_L \omega_E^1 = \int_S d\omega_E^1 \rightarrow \int_S d\omega_E^1 + \frac{\partial \omega_B^2}{\partial t} = 0$$

This equation is true for all possible choices of *L* and *S*, which means the integrand must be 0. In form language and conventional vector language

$$d\omega_E^1 = -\frac{\partial}{\partial t}\omega_B^2 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$