Physics 604 Electromagnetic Theory I

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

Force and Torque

$$B_k\left(\vec{x}\right) = B_k\left(0\right) + \vec{x}\cdot\vec{\nabla}B_k + \cdots$$

$$F_i = \sum_{jk}\varepsilon_{ijk}\left[B_k\left(0\right)\int J_j\left(\vec{x}'\right)d^3\vec{x}' + \int J_j\left(\vec{x}'\right)\vec{x}'\cdot\vec{\nabla}B_k d^3\vec{x}' + \cdots\right]$$

$$= \sum_{jk}\varepsilon_{ijk}\left(\left(\vec{m}\times\vec{\nabla}\right)_j B_k\left(\vec{x}\right)\right)$$

$$\vec{F} = \left(\vec{m}\times\vec{\nabla}\right)\times\vec{B} = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right) - \vec{m}\left(\vec{\nabla}\cdot\vec{B}\right) = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right)$$

$$\vec{\tau} = \int\vec{x}'\times\left[\vec{J}\times\vec{B}\left(0\right)\right]d^3x'$$

$$\vec{\tau} = \int\left[\left(\vec{x}'\cdot\vec{B}\right)\vec{J} - \left(\vec{x}'\cdot\vec{J}\right)\vec{B}\right]d^3x' = \int\left(\vec{x}'\cdot\vec{B}\right)\vec{J}d^3x'$$

$$= \vec{m}\times\vec{B}\left(0\right)$$

Hyperfine Splitting

$$U = -\vec{m}\cdot\vec{B}$$

$$\vec{B}_{orbital}(0) = \frac{\mu_0}{4\pi}\frac{e\vec{L}}{mr^3}$$

$$H_{hfs} = \frac{\mu_0}{4\pi}\left\{\begin{array}{c} -\dfrac{8\pi}{3}\vec{\mu}_e\cdot\vec{\mu}_N\,\delta(\vec{x}) \\ +\dfrac{1}{r^3}\left[\vec{\mu}_e\cdot\vec{\mu}_N - 3\dfrac{(\vec{x}\cdot\vec{\mu}_e)(\vec{x}\cdot\vec{\mu}_N)}{r^2} - \dfrac{e}{m}\vec{L}\cdot\vec{\mu}_N\right]\end{array}\right\}$$

$$\Delta E = -\frac{\mu_0}{4\pi}\frac{8\pi}{3}\left|\psi(0)\right|^2\left\langle\vec{\mu}_e\cdot\vec{\mu}_N\right\rangle$$

B and H

$$\vec{\nabla}\cdot\vec{B}_{micro} = 0 \rightarrow \vec{\nabla}\cdot\vec{B}_{micro} = 0$$

Macroscopic Magnetization

$$\vec{M}\left(\vec{x}\right) = \sum_i n_i \left\langle \vec{m}_i \right\rangle$$

$$\Delta\vec{A}\left(\vec{x}\right) = \frac{\mu_0}{4\pi}\left[\frac{\vec{J}_{free}\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|} + \frac{\vec{M}\left(\vec{x}'\right)\times\left(\vec{x}-\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|^3}\right]$$

$$\int \frac{\vec{M}\left(\vec{x}'\right)\times\left(\vec{x}-\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|^3} d^3\vec{x}' = \int \vec{M}\left(\vec{x}'\right)\times\vec{\nabla}'\left(\frac{1}{\left|\vec{x}-\vec{x}'\right|}\right) d^3\vec{x}'$$

$$\vec{A}\left(\vec{x}\right) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_{free}\left(\vec{x}'\right) + \vec{\nabla}'\times\vec{M}\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|} d^3\vec{x}'$$

Magnetization Current Density

$$\vec{J}_M = \vec{\nabla} \times \vec{M}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J}_{free} + \vec{\nabla} \times \vec{M} \right]$$

Magnetic Field $\quad \vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$

$\vec{\nabla} \times \vec{H} = J_{free} \qquad \vec{\nabla} \times \vec{E} = 0$

$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{D} = \rho_{free}$

isotropic media $\quad \vec{B} = \mu \vec{H}$

paramagnetic $\quad \mu / \mu_0 = 1 +$ a few ppm

diamagnetic $\quad \mu < \mu_0$

ferromagnetic (permanent) $\quad \mu >> \mu_0$

Boundary Conditions

$$\left(\vec{B}_2 - \vec{B}_1\right)\cdot\vec{n} = 0$$

$$\vec{n}\times\left(\vec{H}_2 - \vec{H}_1\right) = K$$

$$\vec{B}_2\cdot\vec{n} = \vec{B}_1\cdot\vec{n} \qquad \vec{B}_2\times\vec{n} = \frac{\mu_2}{\mu_1}\vec{B}_1\times\vec{n}$$

$$\vec{H}_2\cdot\vec{n} = \frac{\mu_1}{\mu_2}\vec{H}_1\cdot\vec{n} \qquad \vec{H}_2\times\vec{n} = \vec{H}_1\times\vec{n}$$

$\mu_1 >> \mu_2$ field almost normal to surface

Magnetostatic Equations

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{H} = \vec{J}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \left(\frac{1}{\mu} \vec{\nabla} \times \vec{A} \right) = \vec{J}$$

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A} = \mu \vec{J} = \frac{\mu}{\mu_0} \vec{J}$$

Current Free

$$\vec{H} = -\vec{\nabla} \Phi_M \qquad \vec{\nabla} \cdot \vec{B}\left[-\vec{\nabla} \Phi_M\right] = 0$$

$$\vec{\nabla} \cdot \left[\mu \vec{\nabla} \Phi_M\right] = 0$$

$$\nabla^2 \Phi_M = 0 \quad \text{for piecewise constant } \mu$$

Hard Ferromagnets

Scalar Potential Solution

$$\vec{\nabla} \cdot \vec{B} = \mu_0 \vec{\nabla} \cdot \left(\vec{H} + \vec{M} \right) = 0$$

$$\nabla^2 \Phi_M = -\rho_M = -\vec{\nabla} \cdot \vec{M}$$

$$\Phi_M\left(\vec{x}\right) = -\frac{1}{4\pi} \int \frac{\vec{\nabla}' \cdot \vec{M}\left(\vec{x}'\right)}{\left|\vec{x} - \vec{x}'\right|} d^3\vec{x}' = \frac{1}{4\pi} \int \vec{M}\left(\vec{x}'\right) \cdot \vec{\nabla}' \left(\frac{1}{\left|\vec{x} - \vec{x}'\right|} \right) d^3\vec{x}'$$

$$\Phi_M\left(\vec{x}\right) = -\frac{1}{4\pi} \vec{\nabla} \cdot \int \frac{\vec{M}\left(\vec{x}'\right)}{\left|\vec{x} - \vec{x}'\right|} d^3\vec{x}'$$

far away

$$\Phi_M\left(\vec{x}\right) = -\frac{1}{4\pi} \vec{\nabla} \left(\frac{1}{r} \right) \cdot \int \vec{M}\left(\vec{x}'\right) d^3\vec{x}' = \frac{\vec{m} \cdot \vec{x}}{r^3}$$

Surface Density

$$\sigma_M = \vec{n} \cdot \vec{M}$$

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' + \frac{1}{4\pi}\oint \frac{\vec{n}' \cdot \vec{M}(\vec{x}')\, da'}{|\vec{x}-\vec{x}'|}$$

Vector Potential Solution

$$\vec{\nabla} \times \vec{H} = \vec{\nabla} \times \left(\vec{B} / \mu_0 - M\right) = 0$$

$$\nabla^2 \vec{A} = -\mu_0 J_M \qquad \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$$

with surface term

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' + \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{x}') \times \vec{n}'}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$$

Uniformly Magnetized Sphere

$$\vec{M} = M_0 \hat{z} \rightarrow \sigma_M = M_0 \cos\theta$$

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int \frac{\vec{\nabla}' \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}' + \frac{1}{4\pi}\oint \frac{d\Omega' \cos\theta}{|\vec{x} - \vec{x}'|}$$

$$\Phi_M(r,\theta) = 0 + \frac{M_0 a^2}{3}\frac{r_<}{r_>^2}\cos\theta$$

inside sphere $\quad \vec{H}_{in} = -\frac{1}{3} \qquad \vec{B}_{in} = \frac{2\mu_0}{3}\vec{M}$

outside sphere $\quad \Phi_M = \frac{M_0 a^3}{3}\frac{\cos\theta}{r^2}$

$$\vec{m} = \frac{4\pi a^3}{3}\vec{M}$$

Alternate Calculations

$$\Phi_M(r,\theta) = -\frac{1}{4\pi}\vec{\nabla}\cdot\int\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3\vec{x}' = -\frac{M_0}{4\pi}\frac{\partial}{\partial z}\int_0^\infty r'^2 dr'\int\frac{d\Omega'}{|\vec{x}-\vec{x}'|}$$

$$= -M_0\cos\theta\frac{\partial}{\partial r}\int_0^a \frac{r'^2}{r_>}dr'$$

with vector potential

$$\vec{M}\times\vec{n}' = M_0\sin\theta'\hat{\phi} = M_0\sin\theta'\left(-\sin\phi'\hat{x}+\cos\phi'\hat{y}\right)$$

$$A_\phi(\vec{x}) = \frac{\mu_0}{4\pi}M_0 a^2\int d\Omega'\frac{\sin\theta'\cos\phi'}{|\vec{x}-\vec{x}'|}$$

$$\sin\theta'\cos\phi' = -\sqrt{\frac{8\pi}{3}}\,\mathrm{Re}\left[Y_{11}(\theta',\phi')\right]$$

$$A_\phi = \frac{\mu_0}{3}M_0 a^2\left(\frac{r_<}{r_>^2}\right)\sin\theta$$

Sphere in External Field

$$\vec{B}_{in} = \vec{B}_0 + \frac{2\mu_0}{3}\vec{M}$$

$$\vec{H}_{in} = \frac{1}{\mu_0}\vec{B}_0 - \frac{1}{3}\vec{M}$$

$$\vec{B}_{in} = \mu \vec{H}_{in}$$

$$\therefore\ \vec{B}_0 + \frac{2\mu_0}{3} = \mu\left(\frac{1}{\mu_0}\vec{B}_0 - \frac{1}{3}\vec{M}\right)$$

$$\vec{M} = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu - 2\mu_0}\right)$$

For ferromagnet use Hysteresis curve plus

$$\vec{B}_{in} + 2\mu_0\vec{H}_{in} = 3\vec{B}_0$$