Physics 604
Electromagnetic
Theory I

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Current Loop

$$J_\phi = I \sin\theta' \delta\left(\cos\theta'\right)\frac{\delta\left(r'-a\right)}{a}$$

$$\vec{J} = J_\phi\left(-\sin\phi'\hat{x}+\cos\phi'\hat{y}\right)$$

$$A_\phi\left(r,\theta\right) = \frac{\mu_0 I}{4\pi a}\int r'^2 dr' d\Omega' \frac{\sin\theta'\cos\phi'\delta\left(r'-a\right)}{\left|\vec{x}-\vec{x}'\right|}$$

$$\left|\vec{x}-\vec{x}'\right| = \left[r^2 + r'^2 - 2rr'\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi'\right)\right]$$

$$A_\phi\left(r,\theta\right) = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}}d\phi'$$

complete elliptical functions

$$A_\phi\left(r,\theta\right) = \frac{\mu_0 I a}{4\pi}\frac{4}{\sqrt{r^2+a^2+2ra\sin\theta}}\left[\frac{\left(2-k^2\right)K\left(k\right)-2E\left(k\right)}{k^2}\right]$$

$$k^2 = \frac{4ar\sin\theta}{r^2+a^2+2ra\sin\theta} \quad B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) \quad B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right) \quad B_\phi = 0$$

Magnetic Dipole Moment

$$A_\phi = \frac{\mu_0 I a^2 r \sin\theta}{4\left(a^2+r^2\right)^{3/2}}\left[1+\frac{15a^2r^2\sin^2\theta}{8\left(a^2+r^2\right)^2}+\cdots\right]$$

$$B_r = \frac{\mu_0 I a^2 \cos\theta}{2\left(a^2+r^2\right)^{3/2}}\left[1+\frac{15a^2r^2\sin^2\theta}{4\left(a^2+r^2\right)^2}+\cdots\right]$$

$$B_\theta = -\frac{\mu_0 I a^2 \sin\theta}{4\left(a^2+r^2\right)^{5/2}}\left[2a^2-r^2+\frac{15a^2r^2\sin^2\theta\left(4a^2-3r^2\right)}{8\left(a^2+r^2\right)^2}+\cdots\right]$$

$$B_r \to \frac{\mu_0}{4\pi}\left(I\pi a^2\right)\frac{2\cos\theta}{r^3} \qquad B_\theta \to \frac{\mu_0}{4\pi}\left(I\pi a^2\right)\frac{\sin\theta}{r^3}$$

magnetic dipole moment $\quad m = I\pi a^2$

Spherical Expansion

$$A_\phi = \frac{\mu_0 I}{a}\mathrm{Re}\sum_{l,m}\frac{Y_{lm}}{2l+1}\int r'^2 dr' d\Omega' \delta(\cos\theta')\delta(r-a)e^{i\phi}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')$$

$$= 2\pi\mu_0 Ia\sum_{l=1}^{\infty}\frac{Y_{l,1}(\theta,0)}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left[Y_{l,1}\left(\frac{\pi}{2},0\right)\right]$$

$$[\ \ ] = \sqrt{\frac{2l+1}{4\pi l(l+1)}}P_l^1(0) = \begin{cases} 0 & l \text{ even} \\ \sqrt{\frac{2l+1}{4\pi l(l+1)}}\left[\frac{(-1)^{n+1}\Gamma(n+3/2)}{\Gamma(n+1)\Gamma(3/2)}\right] & l = 2n+1 \end{cases}$$

$$A_\phi = -\frac{\mu_0 Ia}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n(n+1)!}\frac{r_<^l}{r_>^{l+1}}P_{2n+1}^1(\cos\theta)$$

$$\frac{d}{dx}\left[\sqrt{1-x^2}P_l^1(x)\right] = l(l+1)P_l(x)$$

$$B_r = \frac{\mu_0 Ia}{2r}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n n!}\frac{r_<^{2n+1}}{r_>^{2n+2}}P_{2n+1}(\cos\theta)$$

$$B_\theta = -\frac{\mu_0 Ia^2}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n(n+1)!}\begin{Bmatrix} -\left(\frac{2n+2}{2n+1}\right)\frac{1}{a^3}\left(\frac{r}{a}\right)^{2n} \\ \frac{1}{r^3}\left(\frac{a}{r}\right)^{2n} \end{Bmatrix}P_{2n+1}^1(\cos\theta)$$

Localized Current Distributions

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \cdots$$

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{x}|}\int J_i(\vec{x}')\,d^3x' + \frac{\vec{x}}{|\vec{x}|^3}\cdot\int J_i(\vec{x}')\,\vec{x}'d^3x' + \cdots\right]$$

$$\int\left(f\vec{J}\cdot\vec{\nabla}'g + g\vec{J}\cdot\vec{\nabla}'f + fg\vec{\nabla}'\cdot J\right)d^3x' = 0$$

$$\therefore \int J_i(\vec{x}')\,d^3x' = 0$$

$$\int\left(x_i J_j(\vec{x}') + x_j J_i(\vec{x}')\right)d^3x' = 0$$

Magnetic Moment

$$\vec{x}\cdot\int \vec{x}'J_i d^3x' = \sum_{j=1}^{3} x_j \int x_j' J_i d^3x'$$

$$= -\frac{1}{2}\sum_{i=1}^{3} x_j \int \left(x_i' J_j - x_j' J_i\right) d^3x' = -\frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} x_j \int \left(\vec{x}'\times\vec{J}\right)_k d^3x'$$

$$= -\frac{1}{2}\left[\vec{x}\times\int\left(\vec{x}'\times\vec{J}\right)d^3x'\right]_i$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3} \qquad \vec{\mathrm{M}}(\vec{x}) = \frac{1}{2}\left[\left(\vec{x}\times\vec{J}(\vec{x})\right)\right]$$

magnetic moment

$$\vec{m} = \int \vec{\mathrm{M}}(\vec{x}')d^3\vec{x}' = \frac{1}{2}\int \vec{x}'\times\vec{J}(\vec{x}')d^3\vec{x}'$$

Dipole Field

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi}\left[\frac{3\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}\right]$$ dipole field

current in a plane

$$\vec{m} = \frac{I}{2}\int \vec{x} \times d\vec{l} \qquad |\vec{m}| = I \times (\text{area})$$

relation to angular momentum

$$\vec{J} = \sum_i q_i \vec{\mathrm{v}}_i \delta(\vec{x} - \vec{x}_i)$$

$$\vec{m} = \frac{1}{2}\sum_i q_i \vec{x}_i \times \vec{\mathrm{v}}_i \qquad \vec{L}_i = M_i(\vec{x}_i \times \vec{\mathrm{v}}_i)$$

$$\vec{m} = \sum_i \frac{q_i}{2M_i}\vec{L}_i = \frac{e}{2m_e}\vec{L}_{tot}$$

Correction for Singularity

$$\int_{r<R} \vec{B}(\vec{x})d^3\vec{x} = \int_{r<R} \vec{\nabla}\times\vec{A}d^3\vec{x} = R^2\int_{r<R} d\Omega\,\vec{n}\times\vec{A}$$

$$\int_{r<R} \vec{B}(\vec{x})d^3\vec{x} = -\frac{\mu_0}{4\pi}R^2\int d^3\vec{x}'\vec{J}(\vec{x}')\times\int_{r<R} d\Omega\,\frac{\vec{n}}{|\vec{x}-\vec{x}'|}$$

$$\int_{r<R} \vec{B}(\vec{x})d^3\vec{x} = \frac{\mu_0}{3}\int d^3\vec{x}'\left(\frac{R^2 r_<}{r' r_>^2}\right)\vec{x}'\times\vec{J}(\vec{x}')$$

current inside $\quad \displaystyle\int_{r<R} \vec{B}(\vec{x})d^3\vec{x} = \frac{2\mu_0}{3}\vec{m}$

current outside $\quad \displaystyle\int_{r<R} \vec{B}(\vec{x})d^3\vec{x} = \frac{4\pi R^3}{3}\vec{B}(0)$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\vec{n}(\vec{n}\cdot\vec{m})-\vec{m}}{|\vec{x}|^3}+\frac{8\pi}{3}\vec{m}\delta(\vec{x})\right]$$ dipole field

Force and Torque

$$B_k\left(\vec{x}\right) = B_k\left(0\right) + \vec{x}\cdot\vec{\nabla}B_k + \cdots$$

$$F_i = \sum_{jk}\varepsilon_{ijk}\left[B_k\left(0\right)\int J_j\left(\vec{x}'\right)d^3\vec{x}' + \int J_j\left(\vec{x}'\right)\vec{x}'\cdot\vec{\nabla}B_k d^3\vec{x}' + \cdots\right]$$

$$= \sum_{jk}\varepsilon_{ijk}\left(\left(\vec{m}\times\vec{\nabla}\right)_j B_k\left(\vec{x}\right)\right)$$

$$\vec{F} = \left(\vec{m}\times\vec{\nabla}\right)\times\vec{B} = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right) - \vec{m}\left(\vec{\nabla}\cdot\vec{B}\right) = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right)$$

$$\vec{\tau} = \int\vec{x}'\times\left[\vec{J}\times\vec{B}\left(0\right)\right]d^3x'$$

$$\vec{\tau} = \int\left[\left(\vec{x}'\cdot\vec{B}\right)\vec{J} - \left(\vec{x}'\cdot\vec{J}\right)\vec{B}\right]d^3x' = \int\left(\vec{x}'\cdot\vec{B}\right)\vec{J}d^3x'$$

$$= \vec{m}\times\vec{B}\left(0\right)$$