Physics 604 Electromagnetic Theory I

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Chapt. 5: Magnetostatics and Faraday's Law

magnetic field different than electric field because NO MAGNETIC CHARGES

Fundamental Field is $\vec{B}$, the magnetic induction, or magnetic flux density

$$\Phi_B = \int_S \vec{B} \cdot \vec{n} da \qquad [\Phi_B] = \text{V sec} \quad [B] = \text{V sec/m}^2$$

Direction of $\vec{B}$ direction with no force on an aligned (magnetic) dipole

Magnitude

$$\left|\vec{B}\right| = |\vec{\tau}| / |\mu| \qquad \vec{\mu} \text{ dipole moment vector } [\vec{\mu}] = \text{A m}^2$$

Biot-Savarte Law

Charge conservation $\dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

For magnetostatics $\vec{\nabla} \cdot \vec{J} = 0$

Early Experiments showed

$$d\vec{B} = kI \, \frac{d\vec{l} \times \vec{x}}{\left|\vec{x}\right|^3}$$

$$k = \frac{\mu_0}{4\pi} = 10^{-7} \ \mathrm{N/A}^2$$

for long wire

$$\left|\vec{B}\right| = \frac{\mu_0}{4\pi} IR \int_{-\infty}^{\infty} \frac{dl}{\left(R^2 + l^2\right)^{3/2}} = \frac{\mu_0}{2\pi} \frac{I}{R}$$

Force Between Two Wires

$$d\vec{F} = I_1\left(d\vec{l}_1 \times \vec{B}\right)$$

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{d\vec{l}_1 \times \left(d\vec{l}_2 \times \vec{x}_{12}\right)}{\left|\vec{x}_{12}\right|^3}$$

$$\frac{d\vec{l}_1 \times \left(d\vec{l}_2 \times \vec{x}_{12}\right)}{\left|\vec{x}_{12}\right|^3} = -\left(d\vec{l}_1 \cdot d\vec{l}_2\right)\frac{\vec{x}_{12}}{\left|\vec{x}_{12}\right|^3} + d\vec{l}_2\left(\frac{d\vec{l}_1 \cdot \vec{x}_{12}}{\left|\vec{x}_{12}\right|^3}\right)$$

$$\therefore\ \vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{\left(d\vec{l}_1 \cdot d\vec{l}_2\right)\vec{x}_{12}}{\left|\vec{x}_{12}\right|^3} \qquad \frac{dF}{dl} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$$

$$\vec{F} = \int \vec{J}\left(\vec{x}\right) \times \vec{B}\left(\vec{x}\right) d^3\vec{x} \qquad \vec{\tau} = \int \vec{x} \times \left(\vec{J}\left(\vec{x}\right) \times \vec{B}\left(\vec{x}\right)\right) d^3\vec{x}$$

Differential Equation for B

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}') \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} d^3\vec{x}' = \frac{\mu_0}{4\pi}\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$$

$$\rightarrow \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\vec{\nabla} \times \vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\int \vec{J}(\vec{x}') \cdot \vec{\nabla} \frac{1}{|\vec{x}-\vec{x}'|} d^3\vec{x}' - \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$$

$$= -\frac{\mu_0}{4\pi}\vec{\nabla}\int \vec{J}(\vec{x}') \cdot \vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|} d^3\vec{x}' + \mu_0 \vec{J}(\vec{x})$$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\int \frac{\vec{\nabla}' \cdot \vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' + \mu_0 \vec{J}(\vec{x}) = \mu_0 \vec{J}(\vec{x})$$

Integral Form: Ampere's Law

$$\int_S \vec{\nabla} \times \vec{B} \cdot \vec{n} da = \mu_0 \int_S \vec{J} \cdot \vec{n} da$$

$$\int_L \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{J} \cdot \vec{n} da = \mu_0 I$$

for current free regions

$$\vec{\nabla} \times \vec{B} = 0 \rightarrow \vec{B} = -\vec{\nabla} \Phi_M$$

and the "magnetic potential" solves Laplace Eqn.

For other cases introduce Vector Potential

$$\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x})$$

Vector Potential: Gauge Choice

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}' + \vec{\nabla}\psi$$

ψ gauge function, doesn't matter for $\vec{B}$

free to choose for convenience of solution

$\vec{A} \rightarrow \vec{A} + \nabla\psi$ leaves $\vec{B}$ invariant: gauge transformation

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 \vec{J}$$

$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{J}$ gauge choice $\vec{\nabla} \cdot \vec{A} = 0 \rightarrow \psi$ constant

$$A(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3\vec{x}'$$

Current Loop

$$J_\phi = I \sin\theta' \delta\left(\cos\theta'\right)\frac{\delta\left(r'-a\right)}{a}$$

$$\vec{J} = J_\phi\left(-\sin\phi'\hat{x} + \cos\phi'\hat{y}\right)$$

$$A_\phi\left(r,\theta\right) = \frac{\mu_0 I}{4\pi a}\int r'^2 dr' d\Omega' \frac{\sin\theta' \cos\phi' \delta\left(r'-a\right)}{\left|\vec{x}-\vec{x}'\right|}$$

$$\left|\vec{x}-\vec{x}'\right| = \left[r^2 + r'^2 - 2rr'\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi'\right)\right]$$

$$A_\phi\left(r,\theta\right) = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}}d\phi'$$

complete elliptical functions

$$A_\phi\left(r,\theta\right) = \frac{\mu_0 I a}{4\pi}\frac{4}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}}\left[\frac{\left(2-k^2\right)K\left(k\right)-2E\left(k\right)}{k^2}\right]$$

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) \quad B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right) \quad B_\phi = 0$$