Physics 604
Electromagnetic
Theory I

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Chapter 4: Multipoles and Dielectrics

$$\Phi\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}\left(\theta,\phi\right)}{r^{l+1}}$$

$$\Phi\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|}d^3\vec{x}'$$

$$\Phi\left(\vec{x}\right) = \frac{1}{\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\int Y_{lm}\left(\theta',\phi'\right)r'^{l}\rho\left(\vec{x}'\right)d^3\vec{x}'\right]\frac{Y_{lm}\left(\theta,\phi\right)}{r^{l+1}}$$

$$\therefore\ q_{lm} = Y_{lm}\left(\theta',\phi'\right)r'^{l}\rho\left(\vec{x}'\right)d^3\vec{x}'$$

$$q_{00} = \frac{q}{\sqrt{4\pi}}$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\left(p_x - ip_y\right) \qquad q_{10} = -\sqrt{\frac{3}{4\pi}}p_z$$

Dipole Moment; Quadrupole Moment

$$q_{22} = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\left(Q_{11} - 2iQ_{12} - Q_{22}\right)$$

$$q_{21} = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\left(Q_{13} - iQ_{23}\right)$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}Q_{33}$$

$$q_{l,-m} = (-1)^m q_{lm}^*$$

$$\vec{p} = \int \vec{x}'\rho(\vec{x}')d^3x'$$

$$Q_{ij} = \int \left(3x_i'x_j' - r'^2\delta_{ij}\right)\rho(\vec{x}')\,d^3x'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\,\frac{x_i x_j}{r^5} + \cdots\right]$$

Dipole Field

$$E_r = \frac{(l+1)}{(2l+1)\varepsilon_0} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+2}}$$

$$E_\theta = -\frac{1}{(2l+1)\varepsilon_0} q_{lm} \frac{1}{r^{l+2}} \frac{\partial}{\partial\theta} Y_{lm}(\theta,\phi)$$

$$E_\phi = -\frac{1}{(2l+1)\varepsilon_0} q_{lm} \frac{1}{r^{l+2}} \frac{im}{\sin\theta} Y_{lm}(\theta,\phi)$$

for a dipole in the z direction

$$E_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3} \quad E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} \quad E_\phi = 0$$

$$\vec{E}(\vec{x}) = \frac{3\vec{n}(\vec{n}\cdot\vec{p}) - \vec{p}}{4\pi\varepsilon_0 |\vec{x}-\vec{x}_0|^3}$$

Real Distribution

$$\int_{r<R} \vec{E}(\vec{x}) d^3x = -\int_{r<R} \vec{\nabla}\Phi d^3x = -\int_{r<R} R^2\Phi(\vec{x})\vec{n}\, d\Omega$$

$$= -\frac{R^2}{4\pi\varepsilon_0}\int d^3x' \rho(\vec{x}') \int_{r=R} d\Omega \frac{\vec{n}}{|\vec{x}-\vec{x}'|}$$

$$\vec{n} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$$

$$\int_{r=R} d\Omega \frac{\vec{n}}{|\vec{x}-\vec{x}'|} = \frac{r_<}{r_>^2}\int d\Omega\, \vec{n}\cos\gamma$$

Modified Field

$$\int_{r<R} \vec{E}(\vec{x})d^3x = -\frac{R^2}{3\varepsilon_0}\int d^3x' \frac{r_<}{r_>^2}\rho(\vec{x}')\vec{n}'$$

for all charge inside

$$\int_{r<R} \vec{E}(\vec{x})d^3x = -\frac{\vec{p}}{3\varepsilon_0}$$

For all charge outside

$$\int_{r<R} \vec{E}(\vec{x})d^3x = -\frac{R^3}{3\varepsilon_0}\int d^3x' \frac{1}{r'^2}\rho(\vec{x}')\vec{n}' = \frac{4\pi}{3}R^3\vec{E}(0)$$

$$\therefore \vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3\vec{n}(\vec{n}\cdot\vec{p}) - \vec{p}}{|\vec{x}-\vec{x}_0|^3} - \frac{4\pi}{3}\vec{p}\delta(\vec{x}-\vec{x}_0)\right]$$

Work/Energy

$$W = \int d^3x \rho(\vec{x}) \Phi(\vec{x})$$

Taylor expand

$$\Phi(\vec{x}) = \Phi(0) + \vec{x} \cdot \vec{\nabla}\Phi + \frac{1}{2}\sum_{i,j} x_i x_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j}$$

$$W = q\Phi(0) - \vec{p} \cdot \vec{E}(0) - \frac{1}{6}\sum_{i,j} Q_{ij} \frac{\partial E_j}{\partial x_i}(0) + \cdots$$

For two dipoles

$$W = -\vec{p}_1 \cdot \vec{E}_2 = \frac{\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{n} \cdot \vec{p}_1)(\vec{n} \cdot \vec{p}_2)}{4\pi\varepsilon_0 \left|\vec{x} - \vec{x}_0\right|^3}$$

Ponderable Media

$\vec{\nabla} \times \vec{E}_{micro} = 0 \rightarrow \vec{\nabla} \times \vec{E} = 0$ after averaging

Polarization vector (dipole moment/V)

$\vec{P}\left(\vec{x}\right) = \sum_i n_i \left\langle \vec{p}_i \right\rangle \quad n_i$ number density of type i

by superposition

$$\Delta\Phi\left(\vec{x}, \vec{x}'\right) = \frac{1}{4\pi\varepsilon_0}\left[\frac{\rho\left(\vec{x}'\right)}{\left|\vec{x} - \vec{x}'\right|}\Delta V + \frac{\vec{P}\left(\vec{x}'\right)\cdot\left(\vec{x} - \vec{x}'\right)}{\left|\vec{x} - \vec{x}'\right|^3}\Delta V\right]$$

$$\Phi\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\left[\frac{\rho\left(\vec{x}'\right)}{\left|\vec{x} - \vec{x}'\right|} + \vec{P}\left(\vec{x}'\right)\cdot\vec{\nabla}'\frac{1}{\left|\vec{x} - \vec{x}'\right|}\right]$$