Physics 604 Electromagnetic Theory I

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First Principles Expansion

$$\nabla_x^2 G\left(\vec{x},\vec{x}'\right) = -4\pi\delta\left(\vec{x}-\vec{x}'\right)$$

$$\delta\left(\vec{x}-\vec{x}'\right) = \frac{1}{r^2}\delta\left(r-r'\right)\delta\left(\phi-\phi'\right)\delta\left(\cos\theta-\cos\theta'\right)$$

$$= \frac{1}{r^2}\delta\left(r-r'\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right)$$

Now

$$G\left(\vec{x},\vec{x}'\right) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\left(r\,|\,r',\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right)$$

$$A_{lm}\left(r\,|\,r',\theta',\phi'\right) = g_l\left(r,r'\right)Y_{lm}^*\left(\theta',\phi'\right)$$

$$\frac{1}{r}\frac{d^2}{dr^2}\left(r g_l\left(r,r'\right)\right) - \frac{l\left(l+1\right)}{r^2}g_l\left(r,r'\right) = -\frac{4\pi}{r^2}\delta\left(r-r'\right)$$

$$g_l(r,r') = \begin{cases} Ar^l + Br^{-(l+1)} & r < r' \\ A'r^l + B'r^{-(l+1)} & r > r' \end{cases}$$

inside shell

$$g_l(r,r') = \begin{cases} A\left(r^l - \dfrac{a^{2l+1}}{r^{l+1}}\right) & r < r' \\ B'\left(r^{-(l+1)} - \dfrac{r^l}{b^{2l+1}}\right) & r > r' \end{cases}$$

$$g_l(r,r') = C\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

$$\left\{\frac{d}{dr}\left[rg_l(r,r')\right]\right\}_{r'+\varepsilon} - \left\{\frac{d}{dr}\left[rg_l(r,r')\right]\right\}_{r'-\varepsilon} = -\frac{4\pi}{r'}$$

$$\left\{\frac{d}{dr}\left[rg_l\left(r,r'\right)\right]\right\}_{r'+\varepsilon} = C\left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right)\left\{\frac{d}{dr}\left(\frac{1}{r^l} - \frac{r^{l+1}}{b^{2l+1}}\right)\right\}$$

$$= -\frac{C}{r'}\left(1-\frac{a^{2l+1}}{r'^{2l+1}}\right)\left(l+(l+1)\left(\frac{r'}{b}\right)^{2l+1}\right)$$

$$\left\{\frac{d}{dr}\left[rg_l\left(r,r'\right)\right]\right\}_{r'+\varepsilon} = \frac{C}{r'}\left(l+1+l\left(\frac{a}{r'}\right)^{2l+1}\right)\left(1-\frac{r'^{2l+1}}{b^{2l+1}}\right)$$

$$\therefore C = \frac{4\pi}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}$$

$$G_D\left(\vec{x},\vec{x}'\right) = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

Charged Ring in Sphere

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')G_D(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint \Phi(\vec{x}')\frac{\partial G}{\partial n'}da'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')G_D(\vec{x},\vec{x}')\,d^3x'$$

$$\rho(\vec{x}') = \frac{Q}{2\pi a^2}\delta(r'-a)\delta(\cos\theta')$$

$$\Phi(\vec{x}) = \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} P_l(0)\,r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)P_l(\cos\theta)$$

$$P_{2n+1}(0) = 0 \quad P_{2n}(0) = \left[(-1)^n(2n-1)!!\right]/\,2^n n!$$

$$\Phi(\vec{x}) = \frac{Q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}r_<^{2n}\left(\frac{1}{r_>^{2n+1}} - \frac{r_>^{2n}}{b^{4n+1}}\right)P_{2n}(\cos\theta)$$

$b \to \infty$ yields previous result

Green Function Cylindrical Coords.

$$\nabla_x^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') = -\frac{4\pi}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')$$

$$\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(z-z')} \quad \delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}$$

$$G(\vec{x},\vec{x}') = \frac{1}{4\pi^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dk e^{ik(z-z')} e^{im(\phi-\phi')} g_m(k,\rho,\rho')$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \left(k^2 + \frac{m^2}{\rho^2}\right) g_m = -\frac{4\pi}{\rho}\delta(\rho-\rho')$$

$$g_m = \psi_1(k\rho_<)\psi_2(k\rho_>)$$

$$kW[\psi_1,\psi_2] = -\frac{4\pi}{\rho'} \quad \text{BCs } \psi_1 = I_m, \psi_2 = K_m \quad W[I_m, K_m] = -\frac{1}{x}$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dk e^{ik(z-z')} e^{im(\phi-\phi')} I_m(kr_<) K_m(kr_>)$$

$$= \frac{2}{\pi}\int_0^{\infty} dk \cos[k(z-z')] \times \left\{ I_0(kr_<) K_0(kr_>)/2 + \sum_{m=1}^{\infty}\cos[m(\phi-\phi')] I_m(kr_<) K_m(kr_>) \right\}$$

$$\frac{1}{|\vec{x}-\vec{x}'|}=\frac{1}{\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dk e^{ik(z-z')}e^{im(\phi-\phi')}I_m(kr_<)K_m(kr_>)$$

$$=\frac{2}{\pi}\int_0^{\infty} dk\cos\left[k\left(z-z'\right)\right]\times$$

$$\left\{I_0(kr_<)K_0(kr_>)/2+\sum_{m=1}^{\infty}\cos\left[m\left(\phi-\phi'\right)\right]I_m(kr_<)K_m(kr_>)\right\}$$

$$\frac{1}{\sqrt{\rho^2+z^2}}=\frac{2}{\pi}\int_0^{\infty}\cos kzK_0(k\rho)dk$$

$$\therefore K_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\left(\phi-\phi'\right)}\right)=$$

$$I_0(kr_<)K_0(kr_>)+2\sum_{m=1}^{\infty}\cos\left[m\left(\phi-\phi'\right)\right]I_m(kr_<)K_m(kr_>)$$

$$\ln\left(\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\left(\phi-\phi'\right)}}\right)=2\ln\left(\frac{1}{\rho_>}\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left[m\left(\phi-\phi'\right)\right]$$

Eigenfunction Expansions

$$\nabla^2 \psi = s(\vec{x})$$

$$\nabla^2 \psi + \lambda \psi = 0 \quad \text{with BCs}$$

$$\nabla_x^2 G(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}')$$

$$G(\vec{x}, \vec{x}') = \sum_\lambda a_\lambda(\vec{x}')\psi_\lambda(\vec{x})$$

$$a_\lambda(\vec{x}') = 4\pi \frac{\psi_\lambda^*(\vec{x}')}{\lambda} \quad G(\vec{x}, \vec{x}') = 4\pi \sum_\lambda \frac{\psi_\lambda^*(\vec{x}')\psi_\lambda(\vec{x})}{\lambda}$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{2\pi^2} \int d^3k \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{k^2} \quad \text{for free space}$$

$$G(\vec{x}, \vec{x}') = \frac{32}{\pi abc} \times$$

$$\sum_{l,m,n=1}^{\infty} \frac{\sin(l\pi x/a)\sin(l\pi x'/a)\sin(m\pi y/a)\sin(m\pi y'/a)\sin(n\pi z/a)\sin(l\pi z'/a)}{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}$$

for a box