Physics 604 Electromagnetic Theory I

G. A. Krafft

Jefferson Lab

Jefferson Lab Professor of Physics

Old Dominion University

Zeros

$$J_\nu\left(x_{\nu n}\right)=0 \qquad \left(n=1,2,3,\cdots\right)$$

$$\nu=0 \qquad x_{0n}=2.405, 5.520, \ 8.654, \ldots$$

$$\nu=1 \qquad x_{0n}=3.832, 7.016, 10.173, \ldots$$

$$\nu=2 \qquad x_{0n}=5.136, 8.417, 11.620, \ldots$$

$$x_{\nu n}\approx n\pi+\left(\nu-1/2\right)\frac{\pi}{2}$$

Orthogonality

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_\nu\left(x_{\nu n}\rho/a\right)}{d\rho}\right]+\left(\frac{x_{\nu n}^2}{a^2}+\frac{\nu^2}{\rho^2}\right)J_\nu\left(x_{\nu n}\rho/a\right)=0$$

$$\int_0^a J_\nu\left(x_{\nu n'}\rho/a\right)\left[\rho\frac{dJ_\nu\left(x_{\nu n}\rho/a\right)}{d\rho}\right]d\rho+$$

$$\int_0^a \rho J_\nu\left(x_{\nu n'}\rho/a\right)\left(\frac{x_{\nu n}^2}{a^2}+\frac{\nu^2}{\rho^2}\right)J_\nu\left(x_{\nu n}\rho/a\right)d\rho=0$$

$$-\int_0^a \rho\frac{dJ_\nu\left(x_{\nu n'}\rho/a\right)}{d\rho}\frac{dJ_\nu\left(x_{\nu n}\rho/a\right)}{d\rho}d\rho J_\nu\left(x_{\nu n'}\rho/a\right)$$

$$+\int_0^a \rho J_\nu\left(x_{\nu n'}\rho/a\right)\left(\frac{x_{\nu n}^2}{a^2}+\frac{\nu^2}{\rho^2}\right)J_\nu\left(x_{\nu n}\rho/a\right)d\rho=0$$

$$\left(x_{\nu n}^2 - x_{\nu n'}^2\right)\int_0^a \rho J_\nu\left(x_{\nu n'}\rho/a\right) J_\nu\left(x_{\nu n}\rho/a\right) d\rho = 0$$

$$\int_0^a \rho J_\nu\left(x_{\nu n'}\rho/a\right) J_\nu\left(x_{\nu n}\rho/a\right) d\rho = \frac{a^2}{2}\left[J_{\nu+1}\left(x_{\nu n}\right)\right]^2 \delta_{n'n}$$

Fourier-Bessel Series

$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(x_{\nu n}\rho/a\right)$$

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2\left(x_{\nu n}\right)}\int_0^a \rho f(\rho) J_\nu\left(x_{\nu n}\rho/a\right) d\rho$$

Other Expansion Types

Neumann Series

$$\sum_{n=0}^{\infty} A_n J_{\nu+n}(z)$$

Kapteyn Series (Radiation problems)

$$\sum_{n=0}^{\infty} A_n J_{\nu+n}\left((\nu+n)z\right)$$

Schloemilch Series

$$\sum_{n=0}^{\infty} A_n J_{\nu}(nx)$$

Modified Bessel Functions

$$\frac{1}{x}\frac{d}{dx}\left[x\frac{dR_\nu}{dx}\right]-\left(1+\frac{\nu^2}{x^2}\right)R_\nu=0$$

$$I_\nu=i^{-\nu}J_\nu(ix)\qquad K_\nu=\frac{\pi}{2}i^{\nu+1}H_\nu^{(1)}(ix)$$

$$x<<1\qquad I_\nu(x)\to\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$$

$$K_\nu(x)\to\begin{cases}-\left[\ln\left(\dfrac{x}{2}\right)+0.5772\right] & \nu=0\\ \dfrac{\Gamma(\nu)}{2}\left(\dfrac{2}{x}\right)^\nu & \nu\neq 0\end{cases}$$

$$x>>1,\nu\qquad I_\nu(x)\to\frac{1}{\sqrt{2\pi x}}e^x\left[1+o\left(\frac{1}{x}\right)\right]$$

$$K_\nu(x)\to\frac{1}{\sqrt{2\pi x}}e^{-x}\left[1+o\left(\frac{1}{x}\right)\right]$$

Boundary Value Problem

$$\Phi(\rho,\theta,z)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(k_{mn}\rho)\sinh(k_{mn}z)\left(A_{mn}\sin m\theta+B_{mn}\cos m\theta\right)$$

$$V(\rho,\theta)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(k_{mn}\rho)\sinh(k_{mn}L)\left(A_{mn}\sin m\theta+B_{mn}\cos m\theta\right)$$

$$A_{mn}=\frac{2\cos ech(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)}\int_0^{2\pi}d\theta\int_0^a d\rho\rho V(\rho,\theta)J_m(k_{mn}\rho)\sin m\theta$$

$$B_{mn}=\frac{2\cos ech(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}a)}\int_0^{2\pi}d\theta\int_0^a d\rho\rho V(\rho,\theta)J_m(k_{mn}\rho)\cos m\theta$$

Hankel Transforms

- $a \to \infty$

$$\Phi(\rho,\theta,z) = \sum_{m=0}^{\infty} \int_0^{\infty} dk J_m(k\rho) e^{-kz} \left(A_m(k) \sin m\theta + B_m(k) \cos m\theta \right)$$

$$V(\rho,\theta) = \sum_{m=0}^{\infty} \int_0^{\infty} dk J_m(k\rho) e^{-kz} \left(A_m(k) \sin m\theta + B_m(k) \cos m\theta \right)$$

$$\frac{1}{\pi} \int_0^{2\pi} V(\rho,\theta) \begin{Bmatrix} \sin m\theta \\ \cos m\theta \end{Bmatrix} d\theta = \int_0^{\infty} dk J_m(k\rho) \begin{Bmatrix} A_m(k) \\ B_m(k) \end{Bmatrix}$$

Hankel Transforms

$$\int_0^{\infty} x J_m(kx) J_m(k'x)\, dx = \frac{1}{k} \delta(k - k')$$

$$\left.\begin{matrix} A_m(k) \\ B_m(k) \end{matrix}\right\} = \frac{k}{\pi} \int_0^{\infty} d\rho\rho \int_0^{2\pi} d\theta V(\rho,\theta) J_m(k\rho) \begin{Bmatrix} \sin m\theta \\ \cos m\theta \end{Bmatrix}$$

$$A(x) = \int_0^{\infty} \tilde{A}(k) J_\nu(kx)\, dk \leftrightarrow \tilde{A}(k) = k \int_0^{\infty} x A(x) J_\nu(kx)\, dx$$

Green Function Expansions

$$\frac{1}{\left|\vec{x}-\vec{x}'\right|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right)$$

Exterior Problem

$$G\left(\vec{x},\vec{x}'\right)=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right]Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right)$$

$$\left[\frac{r_<^l}{r_>^{l+1}}-\frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right]=\begin{cases}\dfrac{1}{r'^{l+1}}\left[r^l-\dfrac{a^{2l+1}}{r^l}\right], & r>r'\\[2ex] \left[r'^l-\dfrac{a^{2l+1}}{r'^l}\right]\dfrac{1}{r^{l+1}}, & r'>r\end{cases}$$

First Principles Expansion

$$\nabla_x^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$$

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$$

$$= \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

Now

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}(r\,|\,r',\theta',\phi')Y_{lm}(\theta,\phi)$$

$$A_{lm}(r\,|\,r',\theta',\phi') = g_l(r,r')Y_{lm}^*(\theta',\phi')$$

$$\frac{1}{r}\frac{d^2}{dr^2}\left(rg_l(r,r')\right) - \frac{l(l+1)}{r^2}g_l(r,r') = -\frac{4\pi}{r^2}\delta(r-r')$$

$$g_l(r,r') = \begin{cases} Ar^l + Br^{-(l+1)} & r < r' \\ A'r^l + B'r^{-(l+1)} & r > r' \end{cases}$$

inside shell

$$g_l(r,r') = \begin{cases} A\left(r^l - \dfrac{a^{2l+1}}{r^{-(l+1)}}\right) & r < r' \\ B'\left(r^{-(l+1)} - \dfrac{r^l}{b^{2l+1}}\right) & r > r' \end{cases}$$

$$g_l(r,r') = C\left(r_<^l + \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

$$\left\{\frac{d}{dr}\left[rg_l(r,r')\right]\right\}_{r'+\varepsilon} - \left\{\frac{d}{dr}\left[rg_l(r,r')\right]\right\}_{r'-\varepsilon} = -\frac{4\pi}{r'}$$

$$\left\{\frac{d}{dr}\left[rg_l\left(r,r'\right)\right]\right\}_{r'+\varepsilon} = C\left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right)\left\{\frac{d}{dr}\left(\frac{1}{r^l} - \frac{r^{l+1}}{b^{2l+1}}\right)\right\}$$

$$= -\frac{C}{r'}\left(1 - \frac{a^{2l+1}}{r'^{2l+1}}\right)\left(l + (l+1)\left(\frac{r'}{b}\right)^{2l+1}\right)$$

$$\left\{\frac{d}{dr}\left[rg_l\left(r,r'\right)\right]\right\}_{r'+\varepsilon} = \frac{C}{r'}\left(l + 1 + l\left(\frac{a}{r'}\right)^{2l+1}\right)\left(1 - \frac{r'^{2l+1}}{b^{2l+1}}\right)$$

$$\therefore C = \frac{4\pi}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]}$$

$$G_D\left(\vec{x},\vec{x}'\right) = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}^{*}\left(\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$