Physics 604 Electromagnetic Theory I

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Examples

$$l = 0 \qquad Y_{00} = \frac{1}{4\pi}$$

$$l = 1 \qquad \begin{cases} Y_{11} = -\sqrt{\dfrac{3}{8\pi}} \sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\dfrac{3}{4\pi}} \cos\theta \end{cases}$$

$$l = 2 \qquad \begin{cases} Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}} \sin^2\theta e^{2i\phi} \\ Y_{21} = -\sqrt{\dfrac{15}{8\pi}} \sin\theta\cos\theta e^{i\phi} \\ Y_{20} = \sqrt{\dfrac{5}{4\pi}}\left(\dfrac{3}{2}\cos^2\theta - \dfrac{1}{2}\right) \end{cases}$$

$$l = 3 \qquad \begin{cases} Y_{33} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta e^{3i\phi} \\ Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta e^{2i\phi} \\ Y_{31} = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta\left(5\cos^2\theta - 1\right)e^{i\phi} \\ Y_{20} = \sqrt{\frac{7}{4\pi}}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) \end{cases}$$

$$Y_{l0}\left(\theta,\phi\right) = \sqrt{\frac{2l+1}{4\pi}}P_l\left(\cos\theta\right)$$

Expansions over Sphere

$$g(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}Y_{lm}(\theta,\phi)$$

$$A_{lm}=\int d\Omega Y_{lm}^{*}(\theta,\phi)\,g(\theta,\phi)$$

on the z-axis $(\theta=0)$

$$\left[g(\theta,\phi)\right]=\sum_{l=0}^{\infty}\sqrt{\frac{2l+1}{4\pi}}A_{l0}$$

$$A_{l0}=\sqrt{\frac{2l+1}{4\pi}}\int d\Omega P_l(\cos\theta)\,g(\theta,\phi)$$

General Solution for Laplace Equation

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^{l}+\frac{B_{lm}}{r^{l+1}}\right]Y_{l,m}(\theta,\phi)$$

Addition Theorem

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi)$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

Proof: Because it is a function of (θ,ϕ)

$$P_l(\cos\gamma) = \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} A_{l'm'}(\theta',\phi') Y_{lm}(\theta,\phi)$$

$$\rightarrow A_{l'm'}(\theta',\phi') = \int d\Omega P_l(\cos\theta) Y_{lm}^*(\theta,\phi)$$

$$\nabla'^2 P_l(\cos\gamma) + \frac{l(l+1)}{r^2} P_l(\cos\gamma) = 0$$

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} A_m(\theta',\phi') Y_{lm}(\theta,\phi)$$

$$A_{m'}(\theta',\phi') = \int d\Omega P_l(\cos\theta) Y_{lm}^*(\theta,\phi)$$

$$A_m\left(\theta',\phi'\right)=\frac{4\pi}{2l+1}\left[Y_{lm}^*\left(\theta\left(\gamma,\beta\right),\phi\left(\gamma,\beta\right)\right)\right]_{\gamma=0}$$

alternate expression

$$P_l\left(\cos\gamma\right)=P_l\left(\cos\theta\right)P_l\left(\cos\theta'\right)$$
$$+2\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m\left(\cos\theta\right)P_l^m\left(\cos\theta'\right)\cos\left[m\left(\phi-\phi'\right)\right]$$

Sum Rule

$$\sum_{m=-l}^{l}\left|Y_{lm}\right|^2=\frac{2l+1}{4\pi}$$

General Spherical Expansion

$$\frac{1}{\left|\vec{x}-\vec{x}'\right|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*\left(\theta',\phi'\right)Y_{lm}\left(\theta,\phi\right)$$

Laplace Equation: Cylindrical Coods

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi\left(\rho, \phi, z\right) = R\left(\rho\right) Q\left(\phi\right) Z\left(z\right)$$

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0$$

$$\frac{d^2 Q}{d\phi^2} + \nu^2 Q = 0$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} - \left(k^2 + \frac{\nu^2}{\rho^2}\right) R = 0$$

Bessel Functions

$$Z(z) = e^{\pm kz}$$

$$Q(\phi) = e^{\pm i\nu\phi}$$

Solutions of the radial equation called Bessel Functions

$$R(x) = x^{\alpha} \sum_{j=0}^{\infty} a_j x^j$$

$$\alpha = \pm \nu$$

$$a_{2j} = -\frac{1}{4j(j+\alpha)} a_{2j-2}$$

$$a_{2j} = \frac{(-1)^j \Gamma(\alpha+1)}{2^{2j}\Gamma(j+\alpha+1)}$$

$$J_{\nu}(x) = \left(\frac{x}{2}\right)^{\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

$$J_{-\nu}(x) = \left(\frac{2}{x}\right)^{\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

First kind integer order

$$J_{-m}(x) = (-1)^m J_m(x)$$

$$N_\nu(x) \equiv \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

Hankel Functions

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$$

Recurrence Formulas

$$\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x}\Omega_\nu(x)$$

$$\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = \frac{2\nu}{x}\frac{d\Omega_\nu(x)}{dx}$$

Limiting Behavior

$x << 1$

$$J_\nu(x) \rightarrow \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$$

$$N_\nu(x) \rightarrow \begin{cases} \frac{2}{\pi}\left[\ln\left(\frac{x}{2}\right) + 0.5772\cdots\right], & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases}$$

$x >> 1, \nu$

$$J_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

Zeros

$$J_\nu\left(x_{\nu n}\right) = 0 \qquad \left(n = 1, 2, 3, \cdots\right)$$

$$\nu = 0 \qquad x_{0n} = 2.405, 5.520, \;\; 8.654, \ldots$$

$$\nu = 1 \qquad x_{0n} = 3.832, 7.016, 10.173, \ldots$$

$$\nu = 2 \qquad x_{0n} = 5.136, 8.417, 11.620, \ldots$$

$$x_{\nu n} \approx n\pi + \left(\nu - 1/2\right)\frac{\pi}{2}$$