Physics 604 Electromagnetic Theory I

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Problems with Azimuthal Symmetry

$$\Phi(r,\theta)=\sum_{l=0}^{\infty}\left[A_l r^l+\frac{B_l}{r^{l+1}}\right]P_l(\cos\theta)$$

Potential Inside Hemisphere Problem

$$\Phi(a,\theta)=\sum_{l=0}^{\infty}A_l a^l P_l(\cos\theta)$$

$$A_l=\frac{2l+1}{2a^l}\int_0^{\pi}\Phi(a,\theta)P_l(\cos\theta)\,d\cos\theta$$

$$\Phi(r,\theta)=V\left[\frac{3}{2}\left(\frac{r}{a}\right)P_1(\cos\theta)-\frac{7}{8}\left(\frac{r}{a}\right)^3P_3(\cos\theta)+\frac{11}{16}\left(\frac{r}{a}\right)^5P_5(\cos\theta)+\cdots\right]$$

outside replace $(r/a)^l\rightarrow(a/r)^{l+1}$

Potential Known on Axis

$P_l(1) = 1,$ let symmetry axis be z-axis

$$\Phi(z=r) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] \to \Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$$

$$\Phi(z=r) = V\left[1 - \frac{r^2 - a^2}{r\sqrt{r^2+a^2}}\right] = V\left[1 - \left(1 - \frac{a^2}{r^2}\right)\left(1 - \frac{1}{2}\frac{a^2}{r^2} + \frac{3}{8}\frac{a^4}{r^4} - \frac{15}{48}\frac{a^6}{r^6} + \cdots\right)\right]$$

$$= \frac{V}{\sqrt{\pi}}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{(2j-1/2)\Gamma(j-1/2)}{j!}\left(\frac{a}{r}\right)^{2j}$$

$$\to \Phi(r,\theta) = \frac{V}{\sqrt{\pi}}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{(2j-1/2)\Gamma(j-1/2)}{j!}\left(\frac{a}{r}\right)^{2j} P_l(\cos\theta)$$

$$= V\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \frac{11}{16}\left(\frac{a}{r}\right)^6 P_5(\cos\theta) + \cdots\right]$$

Eqn. 2.27!

Potential for point charge

put $\vec{x}'$ on z-axis

$$\frac{1}{|\vec{x}-\vec{x}'|}=\sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta)$$

$$=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}}=\frac{1}{r-r'}=\frac{1}{r_>\left(1-r_</r_>\right)} \text{ for } \theta=0$$

$$=\frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l$$

$$\therefore\ A_l=B_l=1 \text{ and } \frac{1}{|\vec{x}-\vec{x}'|}=\frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma)\ \ \gamma \text{ angle between}$$

Potential for a Charged Ring

- Ring radius a displaced up by b

$$\Phi\left(z=r\right)=\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+c^2-2cr\cos\alpha}} \qquad c^2=a^2+b^2$$

$r>c$

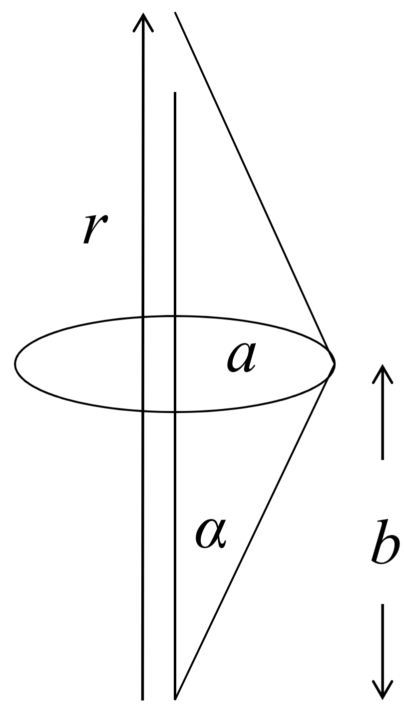
$$\Phi\left(z=r\right)=\frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{c^l}{r^{l+1}}P_l\left(\cos\alpha\right)$$

$r<c$

$$\Phi\left(z=r\right)=\frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{c^{l+1}}P_l\left(\cos\alpha\right)$$

$$\Phi\left(r,\theta\right)=\frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l\left(\cos\alpha\right)P_l\left(\cos\theta\right)$$

$r_<$ is the smaller of r and c and $r_>$ is the larger



Associated Legendre Functions

- What about $m \neq 0$

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_l^m}{dx}\right]+\left[l\left(l+1\right)-\frac{m^2}{1-x^2}\right]P_l^m = 0$$

$$P_l^m\left(x\right)=\left(-1\right)^m\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_l\left(x\right)=\frac{\left(-1\right)^m}{2^l l!}\left(1-x^2\right)^{m/2}\frac{d^{m+l}}{dx^{m+l}}\left(x^2-1\right)^l$$

note $\left|m\right| < l \rightarrow m = -l, -l+1, \cdots, l-1, l$

$$P_l^{-m}\left(x\right)=\left(-1\right)^m\frac{\left(l-m\right)!}{\left(l+m\right)!}P_l^m\left(x\right)$$

orthogonality (note m the same)

$$\int_{-1}^{1} P_l^m\left(x\right)P_{l'}^m\left(x\right)dx = \frac{2}{2l+1}\frac{\left(l-m\right)!}{\left(l+m\right)!}\delta_{ll'}$$

Spherical Harmonics

$$Y_{l,m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)e^{im\phi}$$

$$Y_{l,m}(\theta,\phi) = (-1)^m Y_{l,m}^*(\theta,\phi)$$

orthonormal on sphere

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y_{l,m}^*(\theta,\phi) Y_{l',m'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

completeness

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{l,m}^*(\theta,\phi) Y_{l,m}(\theta',\phi') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$$