Physics 604 Electromagnetic Theory I

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Chapter 3: 3 D Electrostatics

- Laplacian separates in spherical coordinates

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

$$\Phi(\rho,\phi)=\frac{U(r)}{r}P(\theta)Q(\phi)$$

$$PQ\frac{d^2U}{dr^2}+\frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\phi^2}$$

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2}+\frac{1}{Pr^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]+\frac{1}{Q}\frac{d^2Q}{d\phi^2}=0$$

$$Q\propto e^{\pm im\phi}\rightarrow m\ \text{integer}$$

Generalized Legendre Equation

define another separation constant $l(l+1)$ to deal with the radial dependence

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

solution for radial dependence

$$U(r) = Ar^{l+1} + Br^{-l}$$

let $x = \cos\theta$ to convert the θ equation to the generalized Legendre equation

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0$$

Azimuthally Symmetrical Solutions

for $m^2 = 0$, must solve

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dP}{dx}\right)+l\left(l+1\right)P=0$$

power series solutions

$$P\left(x\right)=x^{\alpha}\sum_{j=0}^{\infty}a_j x^j$$

$$\sum_{j=0}^{\infty}\left(\alpha+j\right)\left(\alpha+j-1\right)a_j x^{\alpha+j-2}-\sum_{j=0}^{\infty}\begin{bmatrix}\left(\alpha+j\right)\left(\alpha+j+1\right)\\ -l\left(l+1\right)\end{bmatrix}a_j x^{\alpha+j}=0$$

$$a_0\neq 0\rightarrow\alpha\left(\alpha-1\right)=0$$

$$a_1\neq 0\rightarrow\alpha\left(\alpha+1\right)=0$$

$$a_{j+2}=\left[\frac{\left(\alpha+j\right)\left(\alpha+j+1\right)-l\left(l+1\right)}{\left(\alpha+j+1\right)\left(\alpha+j+2\right)}\right]a_j$$

Legendre Polynomials

for finite solution throughout [-1,1] series must terminate after finite number of terms. Highest power for solution corresponding to l is l. Normalize to have unit value at $x = 1$.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) \qquad P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2 - 1\right)^l$$

Orthonormal Basis on [-1,1]

$$\int_{-1}^{1} P_{l'}(x)\left\{\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_l}{dx}\right]+l(l+1)P_l(x)\right\}dx = 0$$

$$\int_{-1}^{1}\left[\left(x^2-1\right)\frac{dP_{l'}}{dx}\frac{dP_l}{dx}(x)+l(l+1)P_{l'}(x)P_l(x)\right]dx = 0$$

$$\therefore \left[l(l+1)-l'(l'+1)\right]\int_{-1}^{1} P_{l'}(x)P_l(x)\,dx = 0$$

$\therefore l \neq l'$, the integral vanishes. Value when $l = l'$?

$$N_l \equiv \int_{-1}^{1} P_l^2(x)\,dx = \frac{1}{2^{2l}(l!)^2}\int_{-1}^{1}\frac{d^l}{dx^l}\left(x^2-1\right)^l\frac{d^l}{dx^l}\left(x^2-1\right)^l dx$$

$$= \frac{(-1)^l}{2^{2l}(l!)^2}\int_{-1}^{1}\left(x^2-1\right)^l\frac{d^{2l}}{dx^{2l}}\left(x^2-1\right)^l dx = \frac{(2l)!}{2^{2l}(l!)^2}\int_{-1}^{1}\left(1-x^2\right)^l dx$$

$$\left(1-x^2\right)^l=\left(1-x^2\right)\left(1-x^2\right)^{l-1}=\left(1-x^2\right)^{l-1}+\frac{x}{2l}\frac{d}{dx}\left(1-x^2\right)^l$$

$$N_l=\frac{2l-1}{2l}N_{l-1}+\frac{(2l-1)!}{2^{2l}(l!)^2}\int_{-1}^{1}xd\left[\left(1-x^2\right)^l\right]dx$$

$$N_l=\frac{2l-1}{2l}N_{l-1}-\frac{1}{2l}N_l\rightarrow(2l+1)N_l=(2l-1)N_{l-1}$$

$$\therefore\ N_l=\frac{2}{2l+1}\qquad \text{orthonormal basis is } U_l(x)=\sqrt{\frac{2}{2l+1}}P_l(x)$$

Legendre Function Expansions

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x)$$

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x)\, dx$$

example

$$f(x) = +1 \qquad x > 0$$

$$= -1 \qquad x < 0$$

$$A_l = \frac{2l+1}{2} \left[\int_{0}^{1} P_l(x)\, dx - \int_{-1}^{0} P_l(x)\, dx \right]$$

l odd

$$A_l = (2l+1) \int_{0}^{1} P_l(x)\, dx$$

Recurrence Formulas

- Recurrence formula plus Arfken result gives

$$A_l = \left(-\frac{1}{2}\right)^{(l-1)/2} \frac{(2l+1)(l-2)!!}{2\left(\frac{l+1}{2}\right)!} \qquad f(x) = \frac{3}{2}P_1 - \frac{7}{8}P_3 + \frac{11}{16}P_5 + \cdots$$

- Recurrence formulas (from Rodriguez and diff equ.)

$$\frac{dP_{l+1}}{dx} - \frac{dP_{l-1}}{dx} - (2l+1)P_l = 0$$

$$(l+1)P_{l+1} - (2l+1)xP_l + lP_{l-1} = 0$$

$$\frac{dP_{l+1}}{dx} - x\frac{dP_l}{dx} - (l+1)P_l = 0$$

$$(x^2-1)\frac{dP_l}{dx} - lxP_l + lP_{l-1} = 0$$

$$I_1 = \int_{-1}^{1} x P_l(x) P_{l'}(x)\,dx = \frac{1}{2l+1}\int_{-1}^{1} P_{l'}(x)\left[(l+1)P_{l+1}(x) + lP_{l-1}(x)\right]dx$$

$$= \begin{cases} \dfrac{2(l+1)}{(2l+1)(2l+3)}, & l' = l+1 \\[2ex] \dfrac{2l}{(2l-1)(2l+1)}, & l' = l-1 \end{cases}$$

$$I_2 = \int_{-1}^{1} x^2 P_l(x) P_{l'}(x)\,dx = \begin{cases} \dfrac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)}, & l' = l+2 \\[2ex] \dfrac{2l\left(2l^2+2l-1\right)}{(2l-1)(2l+1)(2l+3)}, & l' = l \end{cases}$$

Rodriguez Derivation

$$A_l = \left(2l+1\right)\int_0^1 P_l\left(x\right)dx = \frac{2l+1}{2^l l!}\frac{d^{l-1}}{dx^{l-1}}\left(x^2-1\right)^l\Big|_0^1$$

$$= -\frac{2l+1}{2^l l!}\frac{d^{l-1}}{dx^{l-1}}\left(x^2-1\right)^l\Big|_0 = -\frac{2l+1}{2^l l!}\frac{d^{l-1}}{dx^{l-1}}\sum_{i=0}^{l}\frac{l!\left(-1\right)^i x^{2l-2i}}{\left(l-i\right)!i!}\Bigg|_0$$

The only i that matters because evaluate at 0

$$2\left(l-1\right) = l-1 \rightarrow i = \left(l+1\right)/2$$

$$A_l = -\frac{2l+1}{2^l}\frac{\left(-1\right)^{(l+1)/2}\left(l-1\right)!}{\left(\frac{l-1}{2}\right)!\left(\frac{l+1}{2}\right)!} = \left(-\frac{1}{2}\right)^{(l-1)/2}\frac{\left(2l+1\right)\left(l-1\right)!}{2^{(l+1)/2}\left(\frac{l-1}{2}\right)!\left(\frac{l+1}{2}\right)!}$$

$$= \left(-\frac{1}{2}\right)^{(l-1)/2}\frac{\left(2l+1\right)\left(l-2\right)!!}{2\left(\frac{l+1}{2}\right)!}$$

Recurrence Derivation

$$\frac{d}{dx}\frac{1}{2^{l+1}(l+1)!}\frac{d^{l+1}}{dx^{l+1}}\left(x^2-1\right)^{l+1}-\frac{d}{dx}\frac{1}{2^{l-1}(l-1)!}\frac{d^{l-1}}{dx^{l-1}}\left(x^2-1\right)^{l-1}$$

$$=\frac{1}{2^l l!}\frac{d^{l+1}}{dx^{l+1}}\left[x\left(x^2-1\right)^l\right]-\frac{1}{2^{l-1}(l-1)!}\frac{d^l}{dx^l}\left(x^2-1\right)^{l-1}$$

$$=\frac{1}{2^l l!}\frac{d^l}{dx^l}\left[\left(x^2-1\right)^l+2lx^2\left(x^2-1\right)^{l-1}\right]-\frac{1}{2^{l-1}(l-1)!}\frac{d^l}{dx^l}\left(x^2-1\right)^{l-1}$$

$$=\frac{1}{2^l l!}\frac{d^l}{dx^l}\left[\left(x^2-1\right)^l+2l\left(x^2-1\right)\left(x^2-1\right)^{l-1}\right]=\frac{2l+1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l$$