Physics 604 Electromagnetic Theory I

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2D Potential between two planes

- Fundamental Solutions

$$X = \sin \alpha_n x \quad Y = e^{\pm \alpha_n y}$$

boundary values imply

$$\Phi(x, y, z) = \sum_{n=1}^{\infty} A_n \sin(\alpha_n x) e^{-\alpha_n y}$$

$$A_n = \frac{2}{a} \int_0^a \Phi(x, y = 0) \sin(n\pi x / a)\, dx$$

- For uniform voltage *V*

$$A_n = \frac{4V}{\pi n} \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \quad \Phi(x, y, z) = \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(\alpha_n x) e^{-\alpha_n y}$$

Summation Trick

- Real part and imaginary part of complex analytic functions must satisfy 2D Laplace equations

$$\Phi(x,y) = \frac{4V}{\pi} \operatorname{Im} \sum_{n \text{ odd}} \frac{1}{n} e^{(in\pi/a)(x+iy)} = \frac{4V}{\pi} \operatorname{Im} \sum_{n \text{ odd}} \frac{Z^n}{n}$$

$$\ln(1+Z) - \ln(1-Z) = 2 \sum_{n \text{ odd}} \frac{Z^n}{n}$$

$$\Phi(x,y) = \frac{2V}{\pi} \operatorname{Im}\left[\ln\left(\frac{1+Z}{1-Z}\right)\right] \qquad \frac{1+Z}{1-Z} = \frac{1-|Z|^2 + 2i \operatorname{Im} Z}{|1-Z|^2}$$

$$\Phi(x,y) = \frac{2V}{\pi} \tan^{-1}\left[\frac{\sin(\pi x/a)}{\sinh(\pi y/a)}\right]$$

2D Radial Separation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

$$\Phi(\rho,\phi)=R(\rho)\Psi(\phi)$$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)=\nu^2,\quad \frac{1}{\Psi}\frac{\partial^2\Psi}{\partial\phi^2}=-\nu^2$$

$$R(\rho)=a\rho^{\nu}+b\rho^{-\nu}$$

$$\Psi(\phi)=e^{\pm i\nu\phi}$$

$$\nu=0$$

$$R(\rho)=a_0+b_0\ln\rho \quad \Psi(\phi)=A_0+B_0\phi$$

General Solution

$$\Phi(\rho,\phi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n)$$

$$+ \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n)$$

for conducting corners with opening angle β

$$\nu = \frac{m\pi}{\beta}$$

$$\Phi(\rho,\phi) = V + \sum_{n=1}^{\infty} a_\nu \rho^{\nu\pi/\beta} \sin(\nu\pi\phi/\beta)$$

$$\approx V + a_1 \rho^{\pi/\beta} \sin(\pi\phi/\beta)$$

Fields and Surface Density

$$E_\rho(\rho,\phi) = -\frac{\partial\Phi}{\partial\rho}(\rho,\phi) \approx -\frac{\pi a_1}{\beta}\rho^{\pi/\beta-1}\sin(\pi\phi/\beta)$$

$$E_\phi(\rho,\phi) = -\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}(\rho,\phi) \approx -\frac{\pi a_1}{\beta}\rho^{\pi/\beta-1}\cos(\pi\phi/\beta)$$

$$\sigma(\rho) \approx \varepsilon_0 E_\phi(\rho;\phi=0,\beta) = -\frac{\varepsilon_0\pi a_1}{\beta}\rho^{\pi/\beta-1}$$

points enhance the field, singularity at $\rho\to 0$

Chapter 3: 3 D Electrostatics

- Laplacian separates in spherical coordinates

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

$$\Phi(\rho,\phi)=\frac{U(r)}{r}P(\theta)Q(\phi)$$

$$PQ\frac{d^2U}{dr^2}+\frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\phi^2}$$

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2}+\frac{1}{Pr^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]+\frac{1}{Q}\frac{d^2Q}{d\phi^2}=0$$

$$Q\propto e^{\pm im\phi}\rightarrow m\ \text{integer}$$

Generalized Legendre Equation

define another separation constant $l(l+1)$ to deal with the radial dependence

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

solution for radial dependence

$$U(r) = Ar^{l+1} + Br^{-l}$$

let $x = \cos\theta$ to convert the θ equation to the generalized Legendre equation

$$\frac{d}{dx}\left[\left(1 - x^2\right)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0$$