Physics 604 Electromagnetic Theory I

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

Orthonormal Basis

- What correspond to basis vectors? An orthonormal basis with

$$\langle u_i \mid u_j \rangle = \int_a^b u_i^*(x) u_j(x)\, dx = \delta_{ij}$$

- Best representation of arbitrary function f

$$f(\xi) \leftrightarrow \sum_{n=1}^{N} a_n u_n(\xi)$$

$$M_N = \int_a^b \left| f(\xi) - \sum_{n=1}^{N} a_n u_n(\xi) \right|^2 d\xi$$

minimum when

$$a_n = \langle u_n \mid f \rangle$$

Completeness

- If error can be made arbitrarily small by including enough terms, the *u*s are said to be a *complete* basis and the representation by a_n *converges in the mean* to $f(\xi)$.
- For a complete basis

$$f(\xi) = \sum_{n=1}^{\infty} a_n u_n(\xi) \to \int_a^b \sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) f(\xi') d\xi'$$

$$\therefore \sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) = \delta(\xi - \xi')$$

- This last equality is the completeness relation for the orthonormal basis
- Finally, there are 3-D basis sets, and $a = -\infty$ and/or $b = \infty$ are allowed

Examples

- Fourier Series on [-a/2,a/2] (also represent periodic functions!)

orthonormal basis set $\sqrt{\frac{2}{a}}\cos\left(\frac{2\pi m x}{a}\right)$ and $\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi m x}{a}\right)$

$$f(x) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty}\left[A_m \cos\left(\frac{2\pi m x}{a}\right) + B_m \sin\left(\frac{2\pi m x}{a}\right)\right]$$

then

$$A_m = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\cos\left(\frac{2\pi m x}{a}\right)dx$$

$$B_m = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\sin\left(\frac{2\pi m x}{a}\right)dx$$

Fourier Integrals

- Start with complex exponential representation

$$\text{orthonormal basis} \quad u_m = \frac{1}{\sqrt{a}} e^{2\pi i m x/a}$$

$$f(x) = \frac{1}{\sqrt{a}} \sum_{m=-\infty}^{\infty} A_m e^{2\pi i m x/a}$$

then

$$A_m = \frac{1}{\sqrt{a}} \int_{-a/2}^{a/2} f(x) e^{2\pi i m x/a} dx$$

Large *a* Limit

- Transform

$$\frac{2\pi m}{a} \to k$$

$$\sum_m \quad \to \int_{-\infty}^{\infty} dm = \frac{a}{2\pi} \int_{-\infty}^{\infty} dk$$

$$A_m \to \frac{2\pi}{a} A(k)$$

Fourier Transform Pair

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk \qquad A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x} dx = \delta(k-k') \qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x-x')$$

Laplace Equation/Rectangular Co's

- In Rectangular Coordinates

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

- Separation of Variables

$$\Phi(x, y, z) = X(x)Y(y)Z(z)$$

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z}{dz^2} = 0$$

each term must be separately constant

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = -\alpha^2 \quad \frac{1}{y(z)}\frac{d^2Y}{dy^2} = -\beta^2 \quad \frac{1}{Z(z)}\frac{d^2Z}{dz^2} = \gamma^2$$

$$\alpha^2 + \beta^2 = \gamma^2 \qquad \Phi = e^{i\alpha x} e^{i\beta y} e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$$

3D Box Problem

- Potential Zero on a box except one surface

$$X = \sin \alpha x \quad Y = \sin \beta y \quad Z = \sinh \gamma x$$

boundary values imply

$$\alpha_n = \frac{n\pi}{a} \quad \beta_m = \frac{m\pi}{b} \quad \gamma_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

$$\therefore \Phi(x, y, z) = \sum_{n,m=1}^{\infty} A_{mn} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)$$

- Using B. C. at $z = c$

$$A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy V(x, y) \sin(\alpha_n x) \sin(\beta_m y)$$

- How do you do the potential inside a box? Superposition