Physics 604 Electromagnetic Theory I

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Surface Density/Dipole Moment

- Potential for grounded sphere in uniform field

$$\Phi = -E_0 \left(r - \frac{a^3}{r^2} \right) \cos\theta$$

- Surface density is

$$\sigma = -\varepsilon_0 \left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 3\varepsilon_0 E_0 \cos\theta$$

Integrates to zero: no difference between grounded and free sphere

- Dipole moment is

$$D = 2q'a' = 2\frac{aQ}{R}\frac{a^2}{R} = 4\pi\varepsilon_0 E_0 a^3$$

Green Function for Sphere

- Second Green Function, for the region outside a sphere of radius a

$$G\left(\vec{x},\vec{x}'\right)=\frac{1}{\left|\vec{x}-\vec{x}'\right|}-\frac{a}{x'\left|\vec{x}-a^2\vec{x}'/x'^2\right|}$$

$$=\frac{1}{\left(x^2+x'^2-2xx'\cos\gamma\right)^{1/2}}-\frac{1}{\left(\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma\right)^{1/2}}$$

- Note symmetry applies
- For potential problem need

$$\left.\frac{\partial G}{\partial n'}\right|_{\vec{x}'\in S}=-\frac{x^2-a^2}{a\left(x^2+a^2-2xa\cos\gamma\right)^{3/2}}$$

Potential outside sphere

- Solution to Laplace with given boundary data

$$\Phi\left(\vec{x}\right)=\frac{1}{4\pi}\int\Phi\left(a,\theta',\phi'\right)\frac{a\left(r^2-a^2\right)}{\left(r^2+a^2-2ra\cos\gamma\right)^{3/2}}d\Omega'$$

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\left(\phi-\phi'\right)$$

n.b. $\cos\gamma=\hat{n}\cdot\hat{n}'$

- Two hemispheres at $\pm V$

$$\Phi\left(\vec{x}\right)=\frac{V}{4\pi}\int_0^{2\pi}d\phi'\left\{\int_0^1 d\cos\theta'-\int_{-1}^0 d\cos\theta'\right\}\frac{a\left(r^2-a^2\right)}{\left(r^2+a^2-2ra\cos\gamma\right)^{3/2}}$$

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\left(\phi-\phi'\right)$$

Hemisphere Problem

- Change of variables

$$\theta' \to \pi - \theta' \qquad \phi' = \phi' + \pi$$

$$\Phi(x,\theta,\phi) = \frac{Va\left(r^2 - a^2\right)}{4\pi}\int_0^{2\pi} d\phi' \left\{\int_0^1 d\cos\theta'\right\}\left[\frac{1}{\left(r^2 + a^2 - 2ra\cos\gamma\right)^{3/2}} - \frac{1}{\left(r^2 + a^2 + 2ra\cos\gamma\right)^{3/2}}\right]$$

- Special case positive z-axis $\cos\gamma = \cos\theta$

$$\Phi(z) = V\left[1 - \frac{z^2 - a^2}{z\sqrt{z^2 + a^2}}\right] \to \frac{3Va^2}{2z^2} \qquad \text{for large } z$$

$$\Phi(r,\theta,\phi) = \frac{Va\left(r^2 - a^2\right)}{4\pi\left(r^2 + a^2\right)^{3/2}} \int_0^{2\pi} d\phi' \left\{ \int_0^1 d\cos\theta' \right\} \left[\frac{1}{(1-\alpha\cos\gamma)^{3/2}} - \frac{1}{(1+\alpha\cos\gamma)^{3/2}} \right]$$

$$\alpha = ar / \left(r^2 + a^2\right)$$

Expand; only odd powers!

$$\int_0^{2\pi} d\phi' \int_0^1 d\cos\theta' \cos\gamma = \pi\cos\theta$$

$$\int_0^{2\pi} d\phi' \int_0^1 d\cos\theta' \cos^3\gamma = \frac{\pi}{4}\cos\theta\left(3 - \cos^2\theta\right)$$

$$\Phi(r,\theta,\phi) = \frac{3Va^2 r\left(r^2 - a^2\right)}{2\left(r^2 + a^2\right)^{5/2}}\cos\theta\left[1 + \frac{35}{24}\frac{a^2 r^2}{\left(a^2 + r^2\right)^2}\left(3 - \cos^2\theta\right) + \cdots\right]$$

$$\Phi(r,\theta,\phi) = \frac{3Va^2}{2r^2}\left[\cos\theta + \frac{7}{12}\frac{a^2}{r^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \cdots\right]$$

Orthogonal Functions

- Powerful technique for a wide variety of problems
- Generalization of scalar product projection into ∞ dimensional function spaces (Hilbert spaces)
- Usual scalar product

$$\vec{\mathrm{v}} = \left(\vec{\mathrm{v}} \cdot \hat{x}\right)\hat{x} + \left(\vec{\mathrm{v}} \cdot \hat{y}\right)\hat{y} + \left(\vec{\mathrm{v}} \cdot \hat{z}\right)\hat{z}$$

- One independent variable function space

$$u(x) = \sum_i \left\langle u_i \mid u(x) \right\rangle u_i(x)$$

where u_i(x) are an *orthonormal basis* on some scalar product in a function space, e.g.

$$\left\langle f \mid g \right\rangle = \int_a^b f^*(x)\, g(x)\, dx$$

Orthonormal Basis

- What correspond to basis vectors? An orthonormal basis with

$$\left\langle u_i \mid u_j \right\rangle = \int_a^b u_i^*(x) u_j(x)\, dx = \delta_{ij}$$

- Best representation of arbitrary function f

$$f(\xi) \leftrightarrow \sum_{n=1}^{N} a_n u_n(\xi)$$

$$M_N = \int_a^b \left| f(\xi) - \sum_{n=1}^{N} a_n u_n(\xi) \right|^2 d\xi$$

minimum when

$$a_n = \left\langle u_n \mid f \right\rangle$$

Completeness

- If error can be made arbitrarily small by including enough terms, the *u*s are said to be a *complete* basis and the representation by a_n *converges in the mean* to $f(\xi)$.
- For a complete basis

$$f(\xi) = \sum_{n=1}^{\infty} a_n u_n(\xi) \rightarrow \int_a^b \sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) f(\xi') d\xi'$$

$$\therefore \sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) = \delta(\xi - \xi')$$

- This last equality is the completeness relation for the orthonormal basis
- Finally, there are 3-D basis sets, and $a = -\infty$ and/or $b = \infty$ are allowed