# Physics 604 Electromagnetic Theory I

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# Surface Force: al la Griffiths

- Surface charge distributes so that there is no net force on the charge (remember the charges don't move!). Let $\vec{E}_{eff}$ be the electric field induced in the conductor holding the charges on the surface. By Gaussian pillbox argument

$$\vec{E}_{eff} - \frac{\sigma}{2\varepsilon_0} = 0 \qquad \text{inside conductor}$$

$$\vec{E}_{eff} + \frac{\sigma}{2\varepsilon_0} = E_0 \qquad \text{outside conductor}$$

- Then the total (unbalanced) force per unit area on the surface charge is then simply

$$\sigma E_{eff} = \frac{\sigma^2}{2\varepsilon_0}$$

because the net force between surface charges is zero!

# Variational Principles

- Poisson Solution "Stationary" w.r.t. the variational integral

$$I\left[\psi\right]=\frac{1}{2}\int_V \vec{\nabla}\psi\cdot\vec{\nabla}\psi\, d^3\vec{x}-\int_V \frac{\rho}{\varepsilon_0}\psi\, d^3\vec{x}$$

$$\delta I \equiv I\left[\psi+\delta\psi\right]-I\left[\psi\right]$$

$$=\int_V \vec{\nabla}\psi\cdot\vec{\nabla}\delta\psi\, d^3\vec{x}-\int_V \frac{\rho}{\varepsilon_0}\delta\psi\, d^3\vec{x}$$

$$=-\int_V \left(\nabla^2\psi\right)\delta\psi\, d^3\vec{x}-\int_V \frac{\rho}{\varepsilon_0}\delta\psi\, d^3\vec{x}+\int_S \delta\psi\,\frac{\partial\psi}{\partial n}da$$

- Provided $\delta\psi$=0 on boundary, $\delta I$=0 if

$$\nabla^2\psi=-\frac{\rho}{\varepsilon_0}$$

# Approximate Solutions

- Choose "trial function" forms that minimize integral

$$\Phi\left(\vec{x}\right) = \Phi\left(\vec{x}; a, b, c, d, \ldots\right)$$

$a, b, c, d, \ldots$ expansion coefficients

- Evaluate functional *I*(a,b,c,d,…)
- Best approximation

$$\partial I / \partial a = \partial I / \partial b = \partial I / \partial c = \partial I / \partial d = \ldots = 0$$

- For Neumann BCs the functional to minimize is

$$I\left[\psi\right] = \frac{1}{2}\int_V \vec{\nabla}\psi \cdot \vec{\nabla}\psi \, d^3\vec{x} - \int_V \frac{\rho}{\varepsilon_0}\psi \, d^3\vec{x} - \int_S \frac{\partial \Phi}{\partial n}\psi \, da$$

# Example

- Uniform density to $r = b$, with vanishing BC at $b$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} = -\frac{\rho_0}{\varepsilon_0} \to \Phi(r) = \frac{\rho_0}{4\varepsilon_0}\left(b^2 - r^2\right)$$

- Trial function form

$$\Phi(r) = \alpha\left(\frac{r}{b}\right) - \alpha + \beta\left(\frac{r}{b}\right)^2 - \beta + \gamma\left(\frac{r}{b}\right)^3 - \gamma \quad \mathrm{BC} \to \alpha = 0$$

$$I[\Phi] = \frac{2\pi}{2}\left[\frac{4\beta^2}{b^4}\frac{b^4}{4} + \frac{12\beta\gamma}{b^5}\frac{b^5}{5} + \frac{9\gamma^2}{b^6}\frac{b^6}{6}\right]$$

$$-\frac{2\pi\rho_0}{\varepsilon_0}\left[\frac{\beta}{b^2}\frac{b^4}{4} + \frac{\gamma}{b^3}\frac{b^5}{5} - \beta\frac{b^2}{2} - \gamma\frac{b^2}{2}\right]$$

$$\frac{\partial I}{\partial \beta}=0,\frac{\partial I}{\partial \gamma}=0\rightarrow -\left[\frac{1}{4}-\frac{1}{2}\right]\frac{b^2\rho_0}{\varepsilon_0}+\frac{12}{10}\gamma+\beta=0$$

$$-\left[\frac{1}{5}-\frac{1}{2}\right]\frac{b^2\rho_0}{\varepsilon_0}+\frac{12}{10}\beta+\frac{3}{2}\gamma=0$$

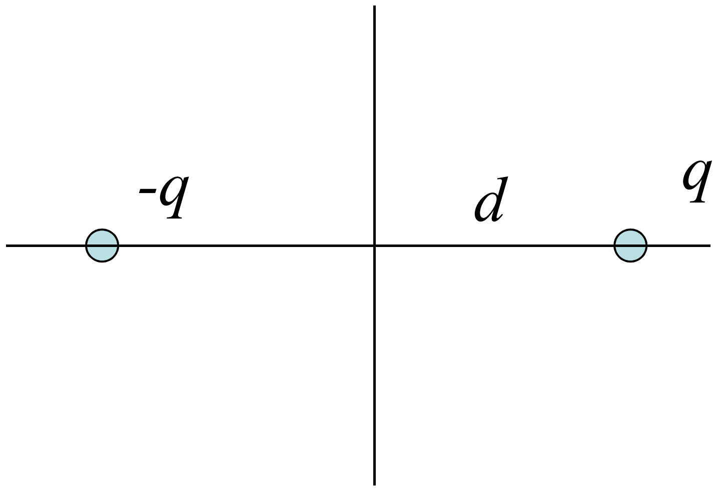
$$\begin{bmatrix} 1 & 12/10 \\ 12/10 & 3/2 \end{bmatrix}\begin{bmatrix} \beta \\ \gamma \end{bmatrix}=\begin{bmatrix} -1/4 \\ -3/10 \end{bmatrix}\frac{\rho_0 b^2}{\varepsilon_0}\rightarrow$$

$$\begin{bmatrix} \beta \\ \gamma \end{bmatrix}=\frac{1}{6/100}\begin{bmatrix} 3/2 & -6/5 \\ -6/5 & 1 \end{bmatrix}\begin{bmatrix} -1/4 \\ -3/10 \end{bmatrix}\frac{\rho_0 b^2}{\varepsilon_0}$$

$$\gamma=0 \quad \beta=-\frac{1}{4}\frac{\rho_0 b^2}{\varepsilon_0}\rightarrow \Phi\left(r\right)=\frac{\rho_0}{4\varepsilon_0}\left(b^2-r^2\right)$$

# Chapter 2

- Method of Images




- By Uniqueness Thm.,

$$\Phi\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left|\vec{x} - d\hat{x}\right|} - \frac{q}{\left|\vec{x} + d\hat{x}\right|}\right] \qquad x > 0$$

is THE solution

# Image for Sphere

- What is the solution inside a corner?
- For a sphere

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} - \frac{q'}{|\vec{x}-\vec{y}'|}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|x\vec{n}-y\vec{n}'|} - \frac{q'}{|x\vec{n}-y'\vec{n}'|}\right]$$

$$\Phi(r=a) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{a|\vec{n}-y\vec{n}'/a|} - \frac{q'}{y'|a\vec{n}/y'-\vec{n}'|}\right]$$

- At $r = a$ cancel when

$$q/a = q'/y' \qquad y/a = a/y' \rightarrow q' = aq/y \qquad y' = a^2/y$$

# Image Force

- Force is attractive

$$\sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial \vec{x}}\bigg|_{x=a} = -\frac{q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1 - a^2/y^2}{\left(1 + a^2/y^2 - 2(a/y)\cos\gamma\right)^{3/2}}$$

$$\left|\vec{F}_{im}\right| = \int_S \frac{\sigma^2}{2\varepsilon_0} da = \frac{q^2}{32\pi^2\varepsilon_0 a^2}\left(\frac{a}{y}\right)^2\left(1 - \frac{a^2}{y^2}\right)^2$$

$$\times \int \frac{\cos\gamma}{\left(1 + a^2/y^2 - 2(a/y)\cos\gamma\right)^3} d\Omega$$

$$\left|\vec{F}_{im}\right| = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\left(1 - \frac{a^2}{y^2}\right)^{-2}$$

# Add Charge to Sphere

- Adding charge to make total charge $Q$ on the sphere get result by superposition

$$\Phi\left(\vec{x}\right)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\left|\vec{x}-\vec{y}\right|}-\frac{aq}{y\left|\vec{x}-\frac{a^2}{y^2}\vec{y}\right|}+\frac{Q+\frac{a}{y}q}{\left|\vec{x}\right|}\right]$$

$$\vec{F}=\frac{1}{4\pi\varepsilon_0}\frac{q}{y^2}\left[Q-\frac{qa^3\left(2y^2-a^2\right)}{y\left(y^2-a^2\right)^2}\right]\frac{\vec{y}}{y}$$

# Sphere at a Certain Potential

- Suppose sphere at potential $V$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} - \frac{aq}{y\left|\vec{x}-\frac{a^2}{y^2}\vec{y}\right|}\right] + \frac{Va}{|\vec{x}|}$$

$$\vec{F}_{im} = \frac{q}{y^2}\left[Va - \frac{1}{4\pi\varepsilon_0}\frac{qay^3}{\left(y^2-a^2\right)^2}\right]\frac{\vec{y}}{y}$$

# Conducting Sphere in Uniform Field

- Suppose $\vec{E} \to E_0 \hat{z}$ at $\pm\infty$. Model uniform field by two charges at large radius.

$$\Phi_{impressed}\left(\vec{x}\right) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\left|\vec{x} + R\hat{z}\right|} - \frac{1}{\left|\vec{x} - R\hat{z}\right|}\right]$$

$$\vec{E}_{impressed}\left(0,0,0\right) = \frac{Q}{2\pi\varepsilon_0}\frac{1}{R^2}\hat{z} \to E_0 = \frac{Q}{2\pi\varepsilon_0}\frac{1}{R^2}$$

- Take limit $Q,R^2 \to \infty$, with a constant ratio to get *uniform* impressed field

$$\Phi(\vec{x}) = \lim_{Q,R^2\to\infty} \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}+R\hat{z}|} - \frac{a}{R\left|\vec{x}+a^2\hat{z}/R\right|} - \frac{1}{|\vec{x}-R\hat{z}|} + \frac{a}{R\left|\vec{x}-a^2\hat{z}/R\right|}\right]$$

$$= \frac{Q/4\pi\varepsilon_0}{\left(r^2+R^2+2rR\cos\theta\right)^{1/2}} - \frac{Q/4\pi\varepsilon_0}{\left(r^2+R^2-2rR\cos\theta\right)^{1/2}}$$

$$- \frac{aQ/4\pi\varepsilon_0}{R\left(r^2+\frac{a^4}{R^2}+\frac{2a^2r}{R}\cos\theta\right)^{1/2}} + \frac{aQ/4\pi\varepsilon_0}{R\left(r^2+\frac{a^4}{R^2}-\frac{2a^2r}{R}\cos\theta\right)^{1/2}}$$

$$\to -\frac{2Q}{4\pi\varepsilon_0 R}\frac{r\cos\theta}{R} + \frac{2aQ}{4\pi\varepsilon_0 Rr}\frac{a^2\cos\theta}{Rr}$$

$$= -\frac{2Q}{4\pi\varepsilon_0 R^2}\left(r-\frac{a^3}{r^2}\right)\cos\theta = -E_0\left(r-\frac{a^3}{r^2}\right)\cos\theta$$

# Surface Density/Dipole Moment

- Surface Density is

$$\sigma = -\varepsilon_0 \left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 3\varepsilon_0 E_0 \cos\theta$$

Integrates to zero: no difference between grounded and free sphere

- Dipole moment is

$$D = 2q'a' = 2\frac{aQ}{R}\frac{a^2}{R} = 4\pi\varepsilon_0 E_0 a^3$$