Physics 604 Electromagnetic Theory I

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Dirichlet Conditions

$$\Phi(\vec{x}) = \int_V \frac{1}{4\pi\varepsilon_0}\rho(\vec{x}')G(\vec{x},\vec{x}')\,d^3x'$$
$$+\frac{1}{4\pi}\int_{S=\partial V}\left(G(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right)da'$$

Demand

$$G_D(\vec{x},\vec{x}') = 0 \qquad\qquad \vec{x}' \in S$$

$$\Phi(\vec{x}) = \int_V \frac{1}{4\pi\varepsilon_0}\rho(\vec{x}')G_D(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\int_{S=\partial V}\Phi(\vec{x}')\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}\,da'$$

n.b., not an integral equation, RHS Φ the boundary values!

Gaussian Pillbox Argument

Green Function Satisfies

$$\nabla'^2 G_D(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \to \frac{\partial G_D}{\partial n'} = -4\pi\delta^2(\vec{x}_t - \vec{x}_t')$$

Integration over surface picks out

$$-\frac{1}{4\pi}\int_{S=\partial V} \Phi(\vec{x}')\frac{\partial G_D(\vec{x},\vec{x}')}{\partial n'}da' \to \Phi(\vec{x})$$

$$G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$$

Neumann Boundary Conditions

$$\Phi\left(\vec{x}\right)=\int_V \frac{1}{4\pi\varepsilon_0}\rho\left(\vec{x}\,'\right)G\left(\vec{x},\vec{x}\,'\right)d^3x'$$

$$+\frac{1}{4\pi}\oint_{S=\partial V}\left(G\left(\vec{x},\vec{x}\,'\right)\frac{\partial\Phi}{\partial n'}-\Phi\left(\vec{x}\,'\right)\frac{\partial G\left(\vec{x},\vec{x}\,'\right)}{\partial n'}\right)da'$$

Demand

$$\frac{\partial G_N}{\partial n'}\left(\vec{x},\vec{x}\,'\right)=0 \qquad \vec{x}\,'\in S$$

not quite because

$$\oint_S \frac{\partial G_N}{\partial n'}da'=-4\pi \qquad \text{so use } \frac{\partial G_N}{\partial n'}\left(\vec{x},\vec{x}\,'\right)=-\frac{4\pi}{S} \qquad \vec{x}\,'\in S$$

$$\Phi\left(\vec{x}\right)=\left\langle\Phi\right\rangle_S+\int_V \frac{1}{4\pi\varepsilon_0}\rho\left(\vec{x}\,'\right)G_N\left(\vec{x},\vec{x}\,'\right)d^3x'+\frac{1}{4\pi}\oint_{S=\partial V}\frac{\partial\Phi}{\partial n'}G_N\left(\vec{x},\vec{x}\,'\right)da'$$

n.b., not an integral equation, RHS Φ the boundary values!

Electric Energy Density

- Work to add a charge from infinity

$$W_i = q_i \Phi\left(\vec{x}_i\right)$$

- To add *n*th charge

$$\Phi\left(\vec{x}_i\right) = \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{\left|\vec{x}_i - \vec{x}_j\right|}$$

$$W_i = \frac{q_i}{4\pi\varepsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{\left|\vec{x}_i - \vec{x}_j\right|}$$

- Total energy

$$W_{tot} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{\left|\vec{x}_i - \vec{x}_j\right|}$$

Symmetric Form

- Omit $i = j$ term in symmetric sum

$$W_{tot} = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{q_i q_j}{\left| \vec{x}_i - \vec{x}_j \right|}$$

- Continuous distribution

$$W = \frac{1}{8\pi\varepsilon_0} \int\int \frac{\rho(\vec{x})\rho(\vec{x}')}{\left| \vec{x} - \vec{x}' \right|} d^3\vec{x} d^3\vec{x}'$$

- In terms of scalar potential

$$W = \frac{1}{2} \int \rho(\vec{x}) \Phi(\vec{x}) d^3\vec{x}$$

Energy Density

- By Poisson Equation

$$W = -\frac{\varepsilon_0}{2}\int\left(\nabla^2\Phi\right)\Phi d^3\vec{x}$$

- Integration by parts

$$W = \frac{\varepsilon_0}{2}\int \vec{\nabla}\Phi\cdot\vec{\nabla}\Phi d^3\vec{x} = \frac{\varepsilon_0}{2}\int \vec{E}\cdot\vec{E} d^3\vec{x}$$

- Energy Density

$$w = \frac{\varepsilon_0}{2}\vec{E}\left(\vec{x}\right)\cdot\vec{E}\left(\vec{x}\right) \qquad \left[\mathrm{J}/\mathrm{m}^3\right]$$

Negative Energy?

- What about two opposite charges?

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_1\left(\vec{x}-\vec{x}_1\right)}{\left|\vec{x}-\vec{x}_1\right|^3} + \frac{1}{4\pi\varepsilon_0}\frac{q_2\left(\vec{x}-\vec{x}_2\right)}{\left|\vec{x}-\vec{x}_2\right|^3}$$

$$32\pi^2\varepsilon_0 w = \frac{q_1^2}{\left|\vec{x}-\vec{x}_1\right|^4} + \frac{q_2^2}{\left|\vec{x}-\vec{x}_2\right|^4} + 2\frac{q_1 q_2\left(\vec{x}-\vec{x}_1\right)\cdot\left(\vec{x}-\vec{x}_2\right)}{\left|\vec{x}-\vec{x}_1\right|^3\left|\vec{x}-\vec{x}_2\right|^3}$$

$$W_{\text{int}} = \frac{q_1 q_2}{16\pi^2\varepsilon_0}\int\frac{\left(\vec{x}-\vec{x}_1\right)\cdot\left(\vec{x}-\vec{x}_2\right)}{\left|\vec{x}-\vec{x}_1\right|^3\left|\vec{x}-\vec{x}_2\right|^3}d^3\vec{x}$$

$$W_{\text{int}} = \frac{q_1 q_2}{4\pi\varepsilon_0\left|\vec{x}_1-\vec{x}_2\right|}\times\frac{1}{4\pi}\int\frac{\vec{\rho}\cdot\left(\vec{\rho}+\vec{n}\right)}{\rho^3\left|\vec{\rho}+\vec{n}\right|^3}d^3\vec{\rho}$$

$$= -\frac{q_1 q_2}{4\pi\varepsilon_0\left|\vec{x}_1-\vec{x}_2\right|}\times\frac{1}{4\pi}\int\frac{\vec{\rho}}{\rho^3}\cdot\nabla_\rho\frac{1}{\left|\vec{\rho}+\vec{n}\right|}d^3\vec{\rho} = \frac{q_1 q_2}{4\pi\varepsilon_0\left|\vec{x}_1-\vec{x}_2\right|}$$

Force on Surface

- Metal-vacuum boundary

$$w = \frac{\varepsilon_0}{2}\left|\vec{E}\right|^2 = \frac{\sigma^2}{2\varepsilon_0}$$

$$F/A = \frac{\sigma^2}{2\varepsilon_0}$$

note factor of 2 (charge

sees 1/2 field on average)

Capacitance

- n conductors with charge placed on them

$$V_i = \sum_{j=1}^{n} p_{ij} Q_j$$

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j$$

$$W = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} V_i V_j$$

Variational Principles

- Poisson Solution "Stationary" w.r.t. the variational integral

$$I\left[\psi\right] = \frac{1}{2}\int_V \vec{\nabla}\psi \cdot \vec{\nabla}\psi \, d^3\vec{x} - \int_V \frac{\rho}{\varepsilon_0}\psi \, d^3\vec{x}$$

$$\delta I \equiv I\left[\psi + \delta\psi\right] - I\left[\psi\right]$$

$$= \int_V \vec{\nabla}\psi \cdot \vec{\nabla}\delta\psi \, d^3\vec{x} - \int_V \frac{\rho}{\varepsilon_0}\delta\psi \, d^3\vec{x}$$

$$= -\int_V \left(\nabla^2\psi\right)\delta\psi \, d^3\vec{x} - \int_V \frac{\rho}{\varepsilon_0}\delta\psi \, d^3\vec{x} + \int_S \delta\psi \frac{\partial\psi}{\partial n} da$$

- Provided $\delta\psi=0$ on boundary, $\delta I=0$ if

$$\nabla^2\psi = -\frac{\rho}{\varepsilon_0}$$

Approximate Solutions

- Choose "trial function" forms that minimize integral

$$\Phi\left(\vec{x}\right) = \Phi\left(\vec{x}; a, b, c, d, \ldots\right)$$

$a, b, c, d, \ldots$ expansion coefficients

- Evaluate functional *I*(a,b,c,d,…)
- Best approximation

$$\partial I / \partial a = \partial I / \partial b = \partial I / \partial c = \partial I / \partial d = \ldots = 0$$

- For Neumann BCs the functional to minimize is

$$I\left[\psi\right] = \frac{1}{2}\int_V \vec{\nabla}\psi \cdot \vec{\nabla}\psi \, d^3\vec{x} - \int_V \frac{\rho}{\varepsilon_0}\psi \, d^3\vec{x} - \int_S \frac{\partial \Phi}{\partial n}\psi \, da$$