Physics 604 Electromagnetic Theory I

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Work

- Will seldom use this expression to find potential: Solve a differential equation!

- Moving a charge q from location $\vec{x}_{before}$ to $\vec{x}_{after}$ requires

$$W = q\left(\Phi\left(\vec{x}_{after}\right) - \Phi\left(\vec{x}_{before}\right)\right) \qquad [\mathrm{eV}]$$

- Falling through a potential difference causes energy gain to a positive charge

- Potential measured in volts [V] or [Nt m/C]

Equations for Potential

- Poisson Equation

$$\nabla^2 \Phi \equiv \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} \right] \Phi = \frac{-\rho}{\varepsilon_0}$$

- Laplace Equation ($\rho = 0$)

$$\nabla^2 \Phi = 0$$

- Verify Solution with vanishing Boundary Conditions at infinity

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} dx'dy'dz'$$

$$\equiv \lim_{a \to 0} \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\rho(\vec{x}')}{\sqrt{(\vec{x} - \vec{x}')^2 + a^2}} dx'dy'dz'$$

- Equation for scalar potential

$$\nabla^2 \Phi \equiv \lim_{a \to 0} \frac{-1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\rho(\vec{x}')\, 3a^2}{\left((\vec{x} - \vec{x}')^2 + a^2 \right)^{5/2}} dx'dy'dz'$$

- Now

$$\lim_{a \to 0} \frac{3a^2}{\left((\vec{x} - \vec{x}')^2 + a^2 \right)^{5/2}} = 0 \qquad \vec{x} \neq \vec{x}'$$

- And

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{3a^2}{\left((\vec{x} - \vec{x}')^2 + a^2 \right)^{5/2}} dx'dy'dz' = 12\pi \int_{0}^{\infty} \frac{x^2}{\left(x^2 + 1 \right)^{5/2}} dx = 4\pi$$

$$\therefore \lim_{a \to 0} \frac{3a^2}{\left((\vec{x} - \vec{x}')^2 + a^2 \right)^{5/2}} = 4\pi\delta(\vec{x} - \vec{x}')$$

Solution to Poisson Equation

$$\nabla^2 \Phi \equiv \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\vec{x}') 4\pi\delta(\vec{x} - \vec{x}')\, dx' dy' dz' = -\frac{\rho}{\varepsilon_0}$$

- Representation

$$\nabla^2 \left[\frac{1}{|\vec{x} - \vec{x}'|} \right] = -4\pi\delta(\vec{x} - \vec{x}')$$

- First Example of a *Green Function*. Made rigorous in the mathematical theory of *distributions*.

Green's Identities

- Green's First Identity

$$\int_V \vec{\nabla} \cdot \vec{A}\, d^3x = \int_{S=\partial V} \vec{A} \cdot n\, da$$

$$\vec{\nabla} \cdot \left(\phi \vec{\nabla} \psi \right) = \phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi$$

$$\phi \vec{\nabla} \psi \cdot \vec{n} \equiv \phi \frac{\partial \psi}{\partial n}$$ is the normal derivative of the field

$$\therefore \int_V \left(\phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi \right) d^3x = \int_{S=\partial V} \phi \frac{\partial \psi}{\partial n}\, da$$

- Green's Second Identity

$$\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi \right) d^3x = \int_{S=\partial V} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) da$$

Formal Solution

$$\phi = \Phi \qquad \psi = 1/R = 1/\left|\vec{x} - \vec{x}'\right|$$

$$\int_V \left(-4\pi\Phi(\vec{x}')\,\delta(\vec{x} - \vec{x}') + \frac{1}{\varepsilon_0 R}\rho(\vec{x}') \right) d^3x'$$

$$= \int_{S=\partial V} \left(\Phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial \Phi}{\partial n'} \right) da'$$

$\vec{x} \in V \rightarrow$

$$\Phi(\vec{x}) = \int_V \frac{1}{4\pi\varepsilon_0 R}\rho(\vec{x}')\,d^3x' + \frac{1}{4\pi}\int_{S=\partial V} \left(\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) \right) da'$$

Surface integral vanishes at ∞ if solutions vanish fast enough.

Surface integral carries the boundary conditions in the solutions!

Uniqueness of Solutions

- Dirichlet Boundary Conditions

$$\text{specify } \phi(\vec{x}) \text{ on a surface, i.e. } \quad \forall \vec{x} \in S$$

- Neumann Boundary Conditions

$$\text{specify } \partial\phi(\vec{x})/\partial n \text{ on a surface, i.e. } \quad \forall \vec{x} \in S$$

- Consider two different solutions Φ_1 and Φ_2, and their difference $U = \Phi_2 - \Phi_1$

$$\int_V \left(U \nabla^2 U + \vec{\nabla} U \cdot \vec{\nabla} U \right) d^3 x = \int_{S=\partial V} U \frac{\partial U}{\partial n} da$$

$$\therefore \int_V \vec{\nabla} U \cdot \vec{\nabla} U \, d^3 x = 0$$

Solution with Specified BCs

- Use Green function method to solve Poisson Equation with specified BCs.

$$\nabla'^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\delta\left(\vec{x}-\vec{x}'\right)$$

Green Function Satisfies

$$\nabla'^2 G\left(\vec{x},\vec{x}'\right) = -4\pi\delta\left(\vec{x}-\vec{x}'\right)$$

but with the boundary conditions of the problem!

$$G\left(\vec{x},\vec{x}'\right) = \frac{1}{|\vec{x}-\vec{x}'|} + F\left(\vec{x},\vec{x}'\right) \qquad \nabla'^2 F\left(\vec{x},\vec{x}'\right) = 0$$

Dirichlet Conditions

$$\Phi\left(\vec{x}\right)=\int_V \frac{1}{4\pi\varepsilon_0}\rho\left(\vec{x}'\right)G\left(\vec{x},\vec{x}'\right)d^3x'$$

$$+\frac{1}{4\pi}\int_{S=\partial V}\left(G\left(\vec{x},\vec{x}'\right)\frac{\partial\Phi}{\partial n'}-\Phi\,\frac{\partial G\left(\vec{x},\vec{x}'\right)}{\partial n'}\right)da'$$

Demand

$$G_D\left(\vec{x},\vec{x}'\right)=0 \qquad\qquad \vec{x}'\in S$$

$$\Phi\left(\vec{x}\right)=\int_V \frac{1}{4\pi\varepsilon_0}\rho\left(\vec{x}'\right)G_D\left(\vec{x},\vec{x}'\right)d^3x'-\frac{1}{4\pi}\int_{S=\partial V}\Phi\,\frac{\partial G_D\left(\vec{x},\vec{x}'\right)}{\partial n'}da'$$