Physics 604 Electromagnetic Theory I

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Electrostatics

- Equations simplify enormously if the fields and sources do not depend on time
- Roughly the first half of this course deals with Static (time independent) Electric fields.
- Roughly the second half of the course will deal with Static Magnetic fields.
- In next semester (Physics 704/804) we'll study dynamic (time dependent phenomena)

- Electric field of a static charge q_1 at location $\vec{x}_1$ given by Coulomb's Law

$$\vec{E}\left(\vec{x}\right) = \frac{q_1}{4\pi\varepsilon_0} \frac{\vec{x} - \vec{x}_1}{\left|\vec{x} - \vec{x}_1\right|^3}$$

Coulomb Interaction

- Notation

$$|\vec{\mathrm{v}}| \equiv \sqrt{\mathrm{v}_x^2 + \mathrm{v}_y^2 + \mathrm{v}_z^2} \rightarrow$$

$$|\vec{x} - \vec{x}_1|^3 = \left((x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2\right)^{3/2}$$

- Force on test particle with charge q at location $\vec{x}$

$$\vec{F}(\vec{x}) = q\vec{E}(\vec{x}) = \frac{qq_1}{4\pi\varepsilon_0} \frac{\vec{x} - \vec{x}_1}{\left((x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2\right)^{3/2}}$$

- Dependence on the distance between charges

$$|\vec{F}| = \frac{qq_1}{4\pi\varepsilon_0} \frac{1}{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} \propto \frac{1}{r^2}$$

Superposition Principal

- Note that likes repel and opposite signs attract
- Field from many charges

$$\vec{E}_{tot}(\vec{x}) = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0} \frac{\vec{x} - \vec{x}_i}{\left|\vec{x} - \vec{x}_i\right|^3}$$

- Example of *SUPERPOSITION*, which arises because LHS of Maxwell Equations are linear differential operators.

$$E_1(\vec{x}) \quad \text{is a solution for M.E. with} \quad \rho_1(\vec{x})$$

$$E_2(\vec{x}) \quad \text{is a solution for M.E. with} \quad \rho_2(\vec{x})$$

$$\downarrow$$

$$E_1(\vec{x}) + E_2(\vec{x}) \quad \text{is a solution for M.E. with} \quad \rho_1(\vec{x}) + \rho_2(\vec{x})$$

Charge Density

- For a continuous static distribution of charge with charge density ρ [C/m^3]

$$\vec{E}(\vec{x}) = \lim_{\Delta x,\Delta y,\Delta z \to 0} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{4\pi\varepsilon_0} \frac{\Delta q(i,j,k)}{\Delta x \Delta y \Delta z} \frac{\vec{x} - \vec{x}_{i,j,k}}{\left|\vec{x} - \vec{x}_{i,j,k}\right|^3} \Delta x \Delta y \Delta z$$

$$\to \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\vec{x}') \frac{(\vec{x} - \vec{x}')}{\left|\vec{x} - \vec{x}'\right|^3} dx' dy' dz'$$

- Recover Coulomb's law for a single charge if distribution for a single charge is Dirac delta-function

$$\rho(\vec{x}) = q_1 \delta(x - x_1) \delta(y - y_1) \delta(z - z_1)$$

$$\to \vec{E}(\vec{x}) = \frac{q_1}{4\pi\varepsilon_0} \frac{\vec{x} - \vec{x}_1}{\left|\vec{x} - \vec{x}_1\right|^3}$$

Properties of Dirac Delta “function”

- Zero-valued

$$\delta\left(x-a\right)=0 \qquad x\neq a$$

- Integrates to 1 if the domain of the integral contains *a*

$$\int_I \delta\left(x-a\right)dx = 1 \qquad a\subset I,$$

$$= 0 \qquad a\not\subset I$$

- Sifting property

$$\int_I f\left(x\right)\delta\left(x-a\right)dx = f\left(a\right) \qquad a\subset I,$$

$$= 0 \qquad a\not\subset I$$

- All domains of integration contain *a* going forward; same pattern applies and get zero if *a* not in the domain

- Derivative property

$$\int f(x)\delta'(x-a)\,dx = -f'(a)$$

- Delta of a function (x_i are the zeros of $f(x)$)

$$\delta\left(f(x)\right) = \sum_i \frac{1}{\left|\frac{df}{dx}(x_i)\right|}\delta(x-x_i)$$

- 3-D delta function

$$\delta\left(\vec{x}-\vec{X}\right) \equiv \delta(x-X)\,\delta(y-Y)\,\delta(z-Z)$$

$$\int_{\Delta V} \delta\left(\vec{x}-\vec{X}\right) d^3x = 1 \quad \vec{X} \subset \Delta V, \ 0 \text{ otherwise}$$

Gauss's Law

- Easy proof:

$$\vec{\nabla}\cdot\frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^3}=\frac{1}{\left|\vec{x}-\vec{x}'\right|^3}-\frac{3\left(x-x'\right)^2}{\left|\vec{x}-\vec{x}'\right|^5}$$

$$+\frac{1}{\left|\vec{x}-\vec{x}'\right|^3}-\frac{3\left(y-y'\right)^2}{\left|\vec{x}-\vec{x}'\right|^5}+\frac{1}{\left|\vec{x}-\vec{x}'\right|^3}-\frac{3\left(z-z'\right)^2}{\left|\vec{x}-\vec{x}'\right|^5}=0 \quad \vec{x}\neq\vec{x}'$$

- Evaluate $\int\left(\vec{x}-\vec{x}'\right)/\left|\vec{x}-\vec{x}'\right|^3\cdot\vec{n}\,dA$ on a sphere surrounding $\vec{x}'$

$$\int_S\frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^3}\cdot\vec{n}\,dA=\frac{1}{R^2}\int R^2 d\cos\theta\, d\phi=4\pi$$

- Independent of *R*! This is it! Do you see it!

Formal Statement

- Gauss's Law: for an *arbitrary* surface S (doesn't need to be a sphere!)

$$\int_S \vec{E} \cdot \vec{n}\, dA = \frac{q_{inside}}{\varepsilon_0}$$

- Pf: If a charge is outside, calculation 1 shows it doesn't contribute to the integral. If a charge is inside, calculation 2 shows it contributes to the integral as given.
- For any V and $S = \partial V$

$$\int_V \nabla \cdot \vec{E}\, dxdydz = \int_S \vec{E} \cdot \vec{n}\, dA = \frac{q_{inside}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_V \rho\left(\vec{x}\right) dxdydz$$

$$\therefore \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

1st Maxwell Eqn. in vacuum

Electrostatic Appoximation

- To fully solve for $\vec{E}\left(\vec{x}\right)$ need equation for $\vec{\nabla}\times\vec{E}$

$$\vec{\nabla}\times\vec{E}\left(\vec{x}\right)=0$$

- Consistent second Maxwell Eqn. when no induction
- Consistent with general expressions for field because

$$\vec{\nabla}\times\frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^3}=0$$

- In Mechanics, such force fields are called conservative because

$$\int_L \vec{E}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{E}\cdot\vec{n}\,dA=0$$

Scalar Potential

- There is a potential energy function, the Scalar Potential with

$$\vec{E} = -\vec{\nabla}\Phi$$

- Is a potential function for particle movement in the electric field
- Large simplification: computing one scalar function easier than three vector component functions
- General expression for scalar potential

$$\frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^3} = -\vec{\nabla}\left(\frac{1}{\left|\vec{x}-\vec{x}'\right|}\right) \to$$

$$\Phi\left(\vec{x}\right) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\rho\left(\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|}dx'dy'dz'$$

Work

- Will seldom use this expression to find potential: Solve a differential equation!
- Moving a charge q from location $\vec{x}_{before}$ to $\vec{x}_{after}$ requires

$$W = q\left(\Phi\left(\vec{x}_{after}\right) - \Phi\left(\vec{x}_{before}\right)\right) \qquad [\mathrm{eV}]$$

- Falling through a potential difference causes energy gain to a positive charge
- Potential measured in volts [V] or [Nt m/C]