Physics 704/804
Mid Term Solution
October 26, 2010

1) a)

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\left(r^{\pm\nu}\cos\nu\theta\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(r^{\pm\nu}\cos\nu\theta\right)$$
$$=\left(\pm\nu\right)^2 r^{\pm\nu-2}\cos\nu\theta-\nu^2 r^{\pm\nu-2}\cos\nu\theta=0$$

b) The potential for the uniform field must satisfy

$$-\frac{\partial\Phi}{\partial x}=E_0\rightarrow\Phi=-E_0x+f\left(y\right)$$
$$-\frac{\partial\Phi}{\partial y}=0\rightarrow f\left(y\right)=C$$

Aside from the physically irrelevant constant, $\Phi\left(r,\theta\right)=-E_0x\left(r,\theta\right)=-E_0r\cos\theta$.

c) Using the same argument as in b), $\Phi\left(r,\theta\right)=-E_0y\left(r,\theta\right)=-E_0r\sin\theta$.

d) In general, the 2-D potential for $r>a$ is

$$\Phi\left(r,\theta\right)=C+\Phi_0\ln r+\sum_{\nu=1}^{\infty}A_\nu r^{\nu}\cos\nu\theta+\sum_{\nu=1}^{\infty}B_\nu r^{\nu}\sin\nu\theta+\sum_{\nu=1}^{\infty}C_\nu r^{-\nu}\cos\nu\theta+\sum_{\nu=1}^{\infty}D_\nu r^{-\nu}\sin\nu\theta$$

The large r behavior implies $\Phi_0=0$, $A_1=-E_0$, the rest of the A_ν and all the B_ν are zero. The boundary condition at $r=a$ implies $C=0$ and

$$\Phi\left(a,\theta\right)=0=-E_0a\cos\theta+\sum_{\nu=1}^{\infty}C_\nu a^{-\nu}\cos\nu\theta+\sum_{\nu=1}^{\infty}D_\nu a^{-\nu}\sin\nu\theta$$

By the orthogonality of the sines and cosines on $\left[-\pi,\pi\right]$ all the D_ν vanish and all the C_ν vanish except

$$C_1=E_0a^2$$
$$\therefore\Phi\left(r,\theta\right)=-E_0\left(r-\frac{a^2}{r}\right)\cos\theta$$

e)

$$\sigma = \varepsilon_0 E_r\left(a,\theta\right) = -\varepsilon_0 \frac{\partial \Phi}{\partial r} = 2\varepsilon_0 E_0 \cos\theta$$

2) a) For the interval $\left[-\pi,\pi\right]$ the orthogonal set is the constant function, $\cos nx$, and $\sin nx$.

Because the function is odd, only the sin's contribute

$$\therefore f\left(x\right) = \sum_{n=1}^{\infty} A_n \sin nx$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(x\right)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} \sin nx dx$$

$$= \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi} = \frac{2}{\pi n}\left(1-\cos n\pi\right)$$

For n even, the coefficient is zero

$$\therefore f\left(x\right) = \frac{4}{\pi}\sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{\sin nx}{n}$$

b) At the value $x = 0.5$, the function has value 1. By a)

$$\sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{\sin nx}{n} = \frac{\pi}{4},$$

for all $x \in \left[0,\pi\right]$.

3) a)

$$E_r = -\frac{\partial \Phi}{\partial r} = \left(l+1\right) A_l \frac{a^{l+1}}{r^{l+2}} P_l\left(\cos\theta\right)$$

$$U = \int_V \frac{\varepsilon_0}{2}\left(l+1\right)^2 A_l^2 \frac{a^{2l+2}}{r^{2l+4}} P_l^2\left(\cos\theta\right) r^2 dr d\cos\theta d\phi$$

$$= \frac{2\pi\varepsilon_0}{2}\left(l+1\right)^2 A_l^2 a^{2l+2}\left[-\frac{1}{\left(2l+1\right)r^{2l+1}}\right]_a^{\infty} \frac{2}{2l+1}$$

$$= 2\pi\varepsilon_0 \frac{\left(l+1\right)^2}{\left(2l+1\right)^2} A_l^2 a$$

b)

$$E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -A_l \frac{a^{l+1}}{r^{l+2}} \frac{dP_l}{d\cos\theta}\left(-\sin\theta\right)$$

$$U = \int_V \frac{\varepsilon_0}{2} A_l^2 \frac{a^{2l+2}}{r^{2l+4}} \frac{dP_l}{d\cos\theta}\left(\sin^2\theta\right)\frac{dP_l}{d\cos\theta} r^2 dr d\cos\theta d\phi$$

$$= \frac{2\pi\varepsilon_0}{2} A_l^2 a^{2l+2}\left[-\frac{1}{(2l+1)r^{2l+1}}\right]\Bigg|_a^\infty \left[-\int_{-1}^{1} \frac{d}{d\cos\theta}\left\{\left(\sin^2\theta\right)\frac{dP_l}{d\cos\theta}\right\} P_l d\cos\theta\right]$$

$$= \frac{2\pi\varepsilon_0}{2} A_l^2 \frac{l(l+1)a}{(2l+1)} \int_{-1}^{1} P_l^2\left(\cos\theta\right) d\cos\theta$$

$$= 2\pi\varepsilon_0 \frac{l(l+1)}{(2l+1)^2} A_l^2 a$$

c) Here the trick is to realize that all the cross terms vanish on integration by the orthonormality of the Legendre polynomials. Therefore the simple formula applies

$$U_{tot} = 2\pi\varepsilon_0 \frac{(l+1)}{(2l+1)} A_l^2 a + 2\pi\varepsilon_0 \frac{(l'+1)}{(2l'+1)} A_{l'}^2 a,$$

where one adds the energy of each term separately.

d) By the usual argument

$$\Phi(r,\theta) = \frac{2\text{ V m}^2}{r^2} P_1\left(\cos\theta\right) + \frac{4\text{ V m}^4}{r^4} P_3\left(\cos\theta\right).$$

Part c) implies

$$U_{tot} = 2\pi\varepsilon_0 \frac{(2)}{(3)} \frac{4\text{ V}^2\text{m}^4}{a^3} + 2\pi\varepsilon_0 \frac{(4)}{(7)} \frac{16\text{ V}^2\text{m}^8}{a^7}$$