Physics 604

Problem Set 6

Due Dec. 09, 2010

1) a) First we need to recall that the current density is

$$\vec{J}(r',\theta',z') = I\delta(r'-a)\delta(z')(\hat{\theta}') = I\delta(r'-a)\delta(z')(-\sin\theta'\hat{x}+\cos\theta'\hat{y})$$

Therefore, by Eqn. 3.148

$$A_\theta(r,\theta,z) = \vec{A}\cdot\hat{\theta} = \frac{\mu_0}{4\pi}\int \frac{I\delta(r'-a)\delta(z')}{|\vec{x}-\vec{x}'|}\cos(\theta-\theta')r'dr'd\theta'dz'$$

$$= \frac{\mu_0}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty\int e^{im(\theta-\theta')}\cos[k(z-z')]I_m(kr_<)K_m(kr_>)I\delta(r'-a)\delta(z')\cos(\theta-\theta')r'dr'd\theta'dz'dk$$

$$= \frac{\mu_0}{2\pi}\int_0^\infty\int \cos[k(z-z')][I_1(kr_<)K_1(kr_>)+I_{-1}(kr_<)K_{-1}(kr_>)]I\delta(r'-a)\delta(z')r'dr'dz'dk$$

$$= \frac{\mu_0}{\pi}\int_0^\infty\int \cos(kz)I_1(kr_<)K_1(kr_>)I\delta(r'-a)r'dr'dk$$

$$= \frac{\mu_0 Ia}{\pi}\int_0^\infty \cos(kz)I_1(kr_<)K_1(kr_>)dk$$

where $r_{\substack{< \\ >}}$ refer to the greater or lesser of a and r.

b) In this part simply use the expansion in Problem Set 4 (Jackson problem 3.16)

$$A_\theta(r,\theta,z) = \frac{\mu_0}{4\pi}\int \frac{I\delta(r'-a)\delta(z')}{|\vec{x}-\vec{x}'|}\cos(\theta-\theta')r'dr'd\theta'dz'$$

$$= \frac{\mu_0}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty\int e^{im(\theta-\theta')}e^{-k(z_>-z_<)}J_m(kr)J_m(kr')I\delta(r'-a)\delta(z')\cos(\theta-\theta')r'dr'd\theta'dz'dk$$

$$= \frac{\mu_0}{4}\int_0^\infty\int e^{-k(z_>-z_<)}[J_1(kr)J_1(kr')+J_{-1}(kr)J_{-1}(kr')]I\delta(r'-a)\delta(z')r'dr'dz'dk$$

$$= \frac{\mu_0}{2}\int_0^\infty\int e^{-k|z|}J_1(kr)J_1(kr')I\delta(r'-a)r'dr'dk$$

$$= \frac{\mu_0 Ia}{2}\int_0^\infty e^{-k|z|}J_1(kr)J_1(ka)dk$$

c) Clearly

$$B_r = -\frac{\partial A_\theta}{\partial z} = \frac{\mu_0 I a}{\pi}\int_0^\infty k\sin(kz) I_1(kr_<) K_1(kr_>)\, dk$$

$$B_z = \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} = \begin{array}{ll} -\dfrac{\mu_0 I a}{\pi}\displaystyle\int_0^\infty k\cos(kz) I_0(kr) K_1(ka)\, dk & r < a \\ -\dfrac{\mu_0 I a}{\pi}\displaystyle\int_0^\infty k\cos(kz) I_1(ka) K_0(kr)\, dk & r > a \end{array}.$$

$$B_r = -\frac{\partial A_\theta}{\partial z} = \begin{array}{ll} \dfrac{\mu_0 I a}{2}\displaystyle\int_0^\infty k e^{-kz} J_1(kr) J_1(ka)\, dk & z > 0 \\ \dfrac{\mu_0 I a}{2}\displaystyle\int_0^\infty -k e^{kz} J_1(kr) J_1(ka)\, dk & z < 0 \end{array}$$

$$B_z = \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} = \frac{\mu_0 I a}{2}\int_0^\infty k e^{-k|z|} J_0(kr) J_1(ka)\, dk$$

where for example we use

$$\frac{1}{r}\frac{\partial(rJ_1(kr))}{\partial r} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial J_0(kr)}{k\partial r}\right) = kJ_0(kr).$$

Evaluating at $r = 0$ implies a non-zero result only for B_z as $J_0(0) = I_0(0) = 1$, and the other functions vanish there. The final results for the integration may be obtained from either 3.150 or problem 3.16. Differentiating inside the integral signs of these expansions yields

$$-\frac{2a}{2\left(a^2+z^2\right)^{3/2}} = \int_0^\infty e^{-k|z|} J_0'(ka)\, kdk = -\int_0^\infty e^{-k|z|} J_1(ka)\, kdk$$

$$-\frac{2a}{2\left(a^2+z^2\right)^{3/2}} = \frac{2}{\pi}\int_0^\infty \cos(kz) K_0'(ka)\, kdk = -\frac{2}{\pi}\int_0^\infty \cos(kz) K_1(ka)\, kdk.$$

Therefore, either case is consistent with the simple Biot-Savart Law calculation

$$B_r(r = 0, z) = 0$$

$$B_z(r = 0, z) = \frac{\mu_0 I a^2}{2\left(a^2+z^2\right)^{3/2}}$$

2) a) By Equation 5.7 in the text, the total force is

$$\vec{F} = I\int d\vec{l} \times \vec{B}.$$

Most answers used an argument like this

$$F_x = I\hat{x}\cdot\int d\vec{l}\times\vec{B} = I\int d\vec{l}\cdot\left(\vec{B}\times\hat{x}\right) \qquad \text{cover formula}$$

$$= I\int\vec{\nabla}\times\left(\vec{B}\times\hat{x}\right)\cdot nda = I\left[\frac{\partial}{\partial x}\vec{B}\right]\cdot n\pi a^2 \qquad \text{cover formula}$$

$$= IB_0\beta n_y\pi a^2 = IB_0\beta\pi a^2\sin\theta_0\sin\phi_0$$

Likewise

$$F_y = B_0\beta\pi a^2\sin\theta_0\cos\phi_0$$

$$\therefore \vec{F} = IB_0\beta\pi a^2\left[\sin\theta_0\sin\phi_0\hat{x} + \sin\theta_0\cos\phi_0\hat{y}\right]$$

Another way is to evaluate the integral directly. The key is to figure out a correct parameterization of the circle. One way is to use two rotation matrices. An equation for the circle is

$$\begin{pmatrix} x(\theta) \\ y(\theta) \\ z(\theta) \end{pmatrix} = \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta_0 & 0 & \sin\theta_0 \\ 0 & 1 & 0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{pmatrix}\begin{pmatrix} a\cos\theta \\ a\sin\theta \\ 0 \end{pmatrix}$$

$$= a\begin{pmatrix} \cos\phi_0\cos\theta_0\cos\theta - \sin\phi_0\sin\theta \\ \sin\phi_0\cos\theta_0\cos\theta + \cos\phi_0\sin\theta \\ -\sin\theta_0\cos\theta \end{pmatrix}$$

Then

$$F_x = Ia\int_0^{2\pi}\begin{pmatrix} \left(-\sin\phi_0\cos\theta_0\sin\theta+\cos\phi_0\cos\theta\right)0 \\ -\sin\theta_0\sin\theta\left(B_0\left(1+\beta a\left(\cos\phi_0\cos\theta_0\cos\theta-\sin\phi_0\sin\theta\right)\right)\right)\end{pmatrix}d\theta$$

$$= I\pi a^2 B_0\beta\sin\theta_0\sin\phi_0$$

$$F_y = Ia\int_0^{2\pi}\begin{pmatrix} \sin\theta_0\sin\theta\left(B_0\left(1+\beta a\left(\sin\phi_0\cos\theta_0\cos\theta+\cos\phi_0\sin\theta\right)\right)\right) \\ -\left(-\cos\phi_0\cos\theta_0\sin\theta-\sin\phi_0\cos\theta\right)0\end{pmatrix}d\theta$$

$$= I\pi a^2 B_0\beta\sin\theta_0\cos\phi_0$$

$$F_z = Ia\int_0^{2\pi}\begin{pmatrix} \left(-\cos\phi_0\cos\theta_0\sin\theta-\sin\phi_0\cos\theta\right)\left(B_0\left(1+\beta a\left(\cos\phi_0\cos\theta_0\cos\theta-\sin\phi_0\sin\theta\right)\right)\right) \\ -\left(-\sin\phi_0\cos\theta_0\sin\theta+\cos\phi_0\cos\theta\right)\left(B_0\left(1+\beta a\left(\sin\phi_0\cos\theta_0\cos\theta+\cos\phi_0\sin\theta\right)\right)\right)\end{pmatrix}d\theta$$

$$= IB_0\beta a^2\int_0^{2\pi}\begin{pmatrix} \left(\sin\phi_0\cos\phi_0\cos\theta_0\sin^2\theta-\sin\phi_0\cos\phi_0\cos\theta_0\cos^2\theta\right) \\ -\left(-\sin\phi_0\cos\phi_0\cos\theta_0\sin^2\theta+\sin\phi_0\cos\phi_0\cos\theta_0\cos^2\theta\right)\end{pmatrix}d\theta$$

$$= 0$$

Note that this expression for the force is exactly consistent with $\vec{F} = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right)$ because

$$\vec{m}\cdot\vec{B} = I\pi a^2 B_0\left(\sin\theta_0\cos\phi_0\left(1+\beta y\right)+\sin\theta_0\cos\phi_0\left(1+\beta x\right)\right)$$
$$\vec{\nabla}\left(\vec{m}\cdot\vec{B}\right) = I\pi a^2 B_0\beta\left(\sin\theta_0\sin\phi_0\hat{x}+\sin\theta_0\cos\phi_0\hat{y}\right).$$

b) Because I have expanded the line integral out, now I can obtain the torque exactly too!

$$\vec{\tau} = \int\vec{x}\times\left(d\vec{l}\times\vec{B}\right)$$
$$= \int_0^{2\pi}\left[\left(\vec{x}\cdot\vec{B}\right)\frac{d\vec{l}}{d\theta}-\left(\vec{x}\cdot\frac{d\vec{l}}{d\theta}\right)\vec{B}\right]d\theta$$

Looking at this expression, non-zero contributions come only from terms that have $\cos^2\theta$ or $\sin^2\theta$ in the expansion. But both $\vec{x}$ and $d\vec{l}$ have single powers of $\cos\theta$ and $\sin\theta$ in their expansion. This means that any term in the magnetic field involving β must integrate to zero in the expansion for the circular loop because of the single powers of $\cos\theta$ and $\sin\theta$ in these terms which integrate to zero on $\cos^2\theta$, $\sin^2\theta$, or $\sin\theta\cos\theta$! Therefore for the circular loop the torque is exactly

$$\vec{\tau} = \vec{m} \times \left(B_0 \hat{x} + B_0 \hat{y} \right)$$

$$\vec{m} = \frac{I}{2} \int \vec{x} \times d\vec{l} = I \pi a^2 \left(\sin\theta_0 \cos\phi_0 \hat{x} + \sin\theta_0 \sin\phi_0 \hat{y} + \cos\theta_0 \hat{z} \right)$$

Of course, non-circular loops would not produce such a simple result. It is an exercise for the reader to show the magnetic moment integrates up in the way indicated.

3) a) For

$$J_z\left(r,\theta\right) = \left(\frac{NI}{2R}\right) \cos\theta \delta\left(r - R\right),$$

there is only a z component of the vector potential A_z . Outside of the current sheet, A_z must solve the 2-D Laplace equation

$$A_z\left(r,\theta\right) = \begin{cases} \sum_{m=-\infty}^{\infty} A_m r^m \cos m\theta & r < a \\ \sum_{m=-\infty}^{\infty} \left[B_m r^m + C_m r^{-m} \right] \cos m\theta & r > a. \end{cases}$$

The possible $\sin m\theta$ terms in the expansion vanish by the form of the current density. If the permeability of the iron is infinite, the radial magnetic induction must vanish at $r = R'$:

$$m B_m R'^{m-1} - m C_m / R'^{m+1} = 0 \rightarrow C_m = B_m R'^{2m}$$

Continuity of the potential function at $r = R$ yields

$$A_m = B_m \left(1 + R'^{2m} / R^{2m}\right)$$

Integrating across the radial singularity in the current density gives

$$r \frac{\partial A_z}{\partial r}\bigg|_{R+\varepsilon} - r \frac{\partial A_z}{\partial r}\bigg|_{R-\varepsilon} = -\mu_0 \left(\frac{NI}{2}\right) \cos\theta$$

$$\left[m B_m \left(R^m - R'^{2m} / R^m\right) - m A_m R^m \right] \cos m\theta = -\mu_0 \left(\frac{NI}{2}\right) \cos\theta$$

Thus $A_m = B_m = 0$ for $m \neq 1$ and

$$A_1 = \mu_0 \left(\frac{NI}{4}\right)\left(\frac{R}{R'^2}\right)\left(1 + \frac{R'^2}{R^2}\right) = \frac{\mu_0}{R}\left(\frac{NI}{4}\right)\left(1 + \frac{R^2}{R'^2}\right)$$

$$B_1 = \mu_0 \left(\frac{NI}{4}\right)\left(\frac{R}{R'^2}\right).$$

The potential only has an $x = r\cos\theta$ dependence. Taking the curl in Cartesian components gives the answer quickest

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\partial A_z}{\partial y}\hat{x} - \frac{\partial A_z}{\partial x}\hat{y} \qquad r < R$$
$$= -\left(\frac{\mu_0 NI}{4R}\right)\left(1+\frac{R^2}{R'^2}\right)\hat{y}.$$

That the sign is correct follows directly from applying the right-hand rule to the original current density.

b) The magnetic energy per unit length inside R is easy to compute because the field is uniform there

$$T'_{inside} = \frac{1}{2\mu_0}\int_0^R\int_0^{2\pi} B^2 r\,dr\,d\theta = \frac{2\pi}{2\mu_0}\frac{\mu_0^2}{R^2}\left(\frac{NI}{4}\right)^2\left(1+\frac{R^2}{R'^2}\right)^2\frac{R^2}{2} \qquad \text{inside } R$$

The magnetic energy per unit length outside is more laborious but straightforward

$$T'_{outside} = \frac{1}{2\mu_0}\int_R^{R'}\int_0^{2\pi}\left[B_r^2 + B_\theta^2\right] r\,dr\,d\theta$$
$$B_r = \frac{\mu_0}{R}\left(\frac{NI}{4}\right)\left(\frac{R^2}{R'^2}\right)\left(1+\frac{R'^2}{r^2}\right)(-\sin\theta)$$
$$B_\theta = -\frac{\mu_0}{R}\left(\frac{NI}{4}\right)\left(\frac{R^2}{R'^2}\right)\left(1-\frac{R'^2}{r^2}\right)(\cos\theta)$$
$$\therefore T'_{outside} = \frac{1}{2\mu_0}\int_R^{R'}\int_0^{2\pi}\left[B_r^2 + B_\theta^2\right] r\,dr\,d\theta$$
$$= \frac{2\pi}{2\mu_0}\frac{\mu_0^2}{R^2}\left(\frac{NI}{4}\right)^2\left(\frac{R^4}{R'^4}\right)\left\{\left[\frac{R'^2}{2}-\frac{R^2}{2}\right]+\frac{R'^4}{-2}\left[\frac{1}{R'^2}-\frac{1}{R^2}\right]\right\}$$
$$= \frac{2\pi}{2\mu_0}\frac{\mu_0^2}{R^2}\left(\frac{NI}{4}\right)^2\left(\frac{R^4}{R'^4}\right)\left[\frac{R'^4}{2R^2}-\frac{R^2}{2}\right]$$

Solving for the potential in Problem 5.30 gives (assuming N current turns)

$$A_1 = \frac{\mu_0}{R}\left(\frac{NI}{4}\right)$$
$$B_1 = 0$$
$$C_1 = \mu_0 R\left(\frac{NI}{4}\right)$$
$$T'_{inside} = \frac{\pi\mu_0}{2}\left(\frac{NI}{4}\right)^2$$
$$T'_{outside} = \pi\mu_0\left(\frac{NI}{4}\right)^2 R^2\int_R^\infty \frac{dr}{r^3} = \frac{\pi\mu_0}{2}\left(\frac{NI}{4}\right)^2$$

The equal partition of the energy without the iron gets shifted so that more field is inside the coil and less field is outside the coil with the iron present.

c) The total magnetic energy per unit length is

$$\begin{aligned}
T'_{total} &= T'_{inside} + T'_{outside} \\
&= \frac{2\pi}{2\mu_0}\frac{\mu_0^2}{R^2}\left(\frac{NI}{4}\right)^2\left[\frac{R^2}{2}+\frac{R^4}{R'^2}+\frac{R^6}{2R'^4}+\frac{R^2}{2}-\frac{R^6}{2R'^4}\right] \\
&= \pi\mu_0\left(\frac{NI}{4}\right)^2\left(1+\frac{R^2}{R'^2}\right)
\end{aligned}$$

The inductance per length is twice the magnetic energy per length divided by the total current squared:

$$\frac{1}{2}L'I^2 = T'$$
$$\therefore L' = \frac{\pi\mu_0 N^2}{8}\left(1+\frac{R^2}{R'^2}\right).$$

4) a) The force between the two current loops is

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi}I_1 I_2 \oint\oint \frac{\left(d\vec{l}_1\cdot d\vec{l}_2\right)\vec{x}_{12}}{\left|\vec{x}_{12}\right|^3}$$

Let the “bar” coordinates of loop 1 be referenced to an origin attached to the loop $\vec{L}_1$, define $\vec{\bar{x}}_1 = \vec{x}_1 - \vec{L}_1$, and similarly for loop 2. Clearly $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2 = \vec{\bar{x}}_1 - \vec{\bar{x}}_2 + \vec{L}_1 - \vec{L}_2 \equiv \vec{\bar{x}}_1 - \vec{\bar{x}}_2 + \vec{R}$ and $\vec{R}$ is the relative displacement of the origins of the loops.

$$
\begin{aligned}
\vec{F}_{12} &= -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{\left(d\vec{l}_1 \cdot d\vec{l}_2\right)\left(\vec{x}_1 - \vec{x}_2 + \vec{R}\right)}{\left|\vec{x}_1 - \vec{x}_2 + \vec{R}\right|^3} \\
&= \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \left(d\vec{l}_1 \cdot d\vec{l}_2\right) \vec{\nabla}_{\vec{R}} \frac{1}{\left|\vec{x}_1 - \vec{x}_2 + \vec{R}\right|} \\
&= \frac{\mu_0}{4\pi} I_1 I_2 \vec{\nabla}_{\vec{R}} \oint\oint \frac{\left(d\vec{l}_1 \cdot d\vec{l}_2\right)}{\left|\vec{x}_1 - \vec{x}_2 + \vec{R}\right|}
\end{aligned}
$$

by differentiating under the integral sign in our well-worn way! Now as long as the two loops don't touch, there is no singularity in the denominator to worry about during the differentiation. Therefore

$$
\begin{aligned}
\vec{F}_{12} &= I_1 I_2 \vec{\nabla}_{\vec{M}} M_{12}\left(\vec{R}\right) \\
M_{12}\left(\vec{R}\right) &= \frac{\mu_0}{4\pi} \oint\oint \frac{\left(d\vec{l}_1 \cdot d\vec{l}_2\right)}{\left|\vec{x}_1 - \vec{x}_2 + \vec{R}\right|}.
\end{aligned}
$$

b) Clearly

$$
\begin{aligned}
\nabla^2_{\vec{R}} M_{12}\left(\vec{R}\right) &= \frac{\mu_0}{4\pi} \oint\oint \left(d\vec{l}_1 \cdot d\vec{l}_2\right) \nabla^2_{\vec{R}} \left[\frac{1}{\left|\vec{x}_1 - \vec{x}_2 + \vec{R}\right|}\right] \\
&= -\frac{\mu_0}{4\pi} \oint\oint \left(d\vec{l}_1 \cdot d\vec{l}_2\right) 4\pi\delta\left(\vec{x}_1 - \vec{x}_2 + \vec{R}\right) = 0,
\end{aligned}
$$

again, as long as the loops do not touch each other.