Physics 604
Problem Set 3
Due Oct. 21, 2010

1) a) The orthogonal expansion functions with the correct boundary conditions in the *x*-direction and *y*-direction are products of the vanishing sines. Therefore the potential must be of the form

$$\Phi(x,y,z)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}e^{k_{mn}z}+B_{mn}e^{-k_{mn}z}\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$k_{mn}=\frac{\pi}{a}\sqrt{m^2+n^2}$$

where the terms with *m*=0 or *n*=0 are excluded because the sines then vanish. Following the example in the text, the boundary conditions at $z=0,a$ yield

$$\left(A_{mn}+B_{mn}\right)\frac{a^2}{4}=\frac{a}{\pi m}\frac{a}{\pi n}\left(\cos 0-\cos m\pi\right)\left(\cos 0-\cos n\pi\right)=V4\frac{a}{\pi m}\frac{a}{\pi n}\qquad m,n\quad\text{odd}$$

$$\left(A_{mn}e^{k_{mn}a}+B_{mn}e^{-k_{mn}a}\right)\frac{a^2}{4}=\frac{a}{\pi m}\frac{a}{\pi n}\left(\cos 0-\cos m\pi\right)\left(\cos 0-\cos n\pi\right)=V4\frac{a}{\pi m}\frac{a}{\pi n}\qquad m,n\quad\text{odd}$$

Solving the linear equations yields

$$A_{mn}=V\frac{16\left(e^{-k_{mn}a}-1\right)}{e^{-k_{mn}a}-e^{k_{mn}a}}\frac{1}{\pi^2 mn}\qquad m,n\quad\text{odd}$$

$$B_{mn}=V\frac{16\left(-e^{k_{mn}a}+1\right)}{e^{-k_{mn}a}-e^{k_{mn}a}}\frac{1}{\pi^2 mn}\qquad m,n\quad\text{odd}$$

and the final expression for the potential is

$$\Phi(x,y,z)=\frac{16V}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left[\sinh\left(k_{mn}z\right)-\sinh\left\{k_{mn}\left(z-a\right)\right\}\right]}{mn\sinh k_{mn}a}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$k_{mn}=\frac{\pi}{a}\sqrt{m^2+n^2}.$$

where the summation is only over odd integers m,n.

b) At $\vec{x}=\left(a/2,a/2,a/2\right)$ the potential evaluates to

$$\Phi\left(a/2,a/2,a/2\right)=\frac{32V}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sinh\left(k_{mn}a/2\right)}{mn\sinh k_{mn}a}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right),$$

where the sine factors simply alternate signs in the various terms. Because the sinh terms grow rapidly with m and n, only the m,n=11,13,31,33,15,51 need to be summed to give a result good to three significant digits.

$$\frac{\sinh\sqrt{2}\pi/2}{\sinh\sqrt{2}\pi}=0.1071818$$
$$\frac{2}{3}\frac{\sinh\sqrt{10}\pi/2}{\sinh\sqrt{10}\pi}=\frac{2\times 0.00691623}{3}=0.0046411$$
$$\frac{1}{9}\frac{\sinh 3\sqrt{2}\pi/2}{\sinh 3\sqrt{2}\pi}=\frac{0.00001483}{9}=0.0000016$$
$$\frac{2}{5}\frac{\sinh\sqrt{26}\pi/2}{\sinh\sqrt{26}\pi}=\frac{2\times 0.00033228}{5}=0.0001329$$

The sum of these terms with the correct signs is 0.1026752. When multiplied by $32/\pi^2$ the sum is 0.3329… . By the problem quoted, the sum "should be" 1/3=0.3333... .

A few of you realized that there was a very compact answer in terms of cosh's for both the potential and the summations. Cute!!!

2) This problem is a straightforward application of the definition of completeness. Assuming completeness of the expansion basis implies

$$f(\phi)=\sum_{m=1}^{\infty}A_m\sin\frac{m\pi\phi}{\beta}\qquad \text{where } A_m=\left\langle \sin\frac{m\pi\phi}{\beta}\middle| f\right\rangle,$$

for suitably well-behaved (mathematicians define this, the "delta-function" is OK) functions $f(\phi)$. In particular,

$$\delta(\phi-\phi')=\sum_{m=1}^{\infty}\delta_m\sin\frac{m\pi\phi}{\beta}\qquad \delta_m=\int_0^{\beta}\delta(\phi-\phi')\sin\frac{m\pi\phi}{\beta}d\phi\Big/\int_0^{\beta}\sin^2\frac{m\pi\phi}{\beta}d\phi,$$

where the division in the expression for the expansion coefficient accounts for the normalization of the orthogonal set. Therefore

$$\delta(\phi-\phi')=\sum_{m=1}^{\infty}\frac{2}{\beta}\sin\frac{m\pi\phi'}{\beta}\sin\frac{m\pi\phi}{\beta}.$$

3) a) This is an example of the 2-D polar coordinate expansion set. In general,

$$\Phi(r,\theta)=\Phi_0\ln r+\sum_{l=1}^{\infty}\left[A_l r^l+B_l r^{-l}\right]\left[C_l\sin\frac{l\pi\theta}{\beta}+D_l\cos\frac{l\pi\theta}{\beta}\right].$$

The boundary conditions on the radial planes $\theta = 0, \beta$ imply only the sin terms appear and C_l may be taken to be 1, and $\Phi_0 = 0$. The boundary condition at $r = a$ implies $B_l = -A_l a^{2l}$, so the expansion is

$$\Phi(r,\theta) = \sum_{l=1}^{\infty} A_l \left[r^l - a^{2l} r^{-l} \right] \sin \frac{l\pi\theta}{\beta}.$$

b) The lowest non-vanishing term has $l = 1$. In this case

$$E_r = -\frac{\partial \Phi}{\partial r} = -A_1 \left[1 + \frac{a^2}{r^2} \right] \sin \frac{\pi\theta}{\beta}$$
$$E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -\frac{\pi}{\beta} A_1 \left[1 - a^2 / r^2 \right] \cos \frac{\pi\theta}{\beta}.$$

The surface densities are

$$\sigma(r) = \frac{E_\theta}{\varepsilon_0} = -\frac{\pi}{\varepsilon_0 \beta} A_1 \left[1 - a^2 / r^2 \right]$$
$$\sigma(\theta) = \frac{E_r}{\varepsilon_0} = -2A_1 \sin \frac{\pi\theta}{\beta}.$$

The negative sign occurs on both the $\theta = 0, \beta$ surfaces.

c) At large radii $r >> a$ and for $\beta = \pi$

$$\vec{E} \to -A_1 \sin\theta \hat{r} - A_1 \cos\theta \hat{\theta} =$$
$$-A_1 \sin\theta \left(\cos\theta \hat{x} + \sin\theta \hat{y} \right) - A_1 \cos\theta \left(-\sin\theta \hat{x} + \cos\theta \hat{y} \right).$$
$$= -A_1 \hat{y}$$

The y-direction is the normal direction. Also, we can interpret the expansion coefficient A_1 as the negative of the normal electric field at large radii. The total charge per unit length on the half-cylinder is

$$q' = \frac{1}{\varepsilon_0} \int_0^\pi -2A_1 a \sin\theta d\theta = \frac{2A_1 a}{\varepsilon_0} \cos\theta \Big|_0^\pi = \frac{-4A_1 a}{\varepsilon_0}$$

In the absence of the the half cylinder, the charge per unit length in a width of length $2a$ would be

$$q' = \frac{-A_1}{\varepsilon_0}(2a)$$

because in this case, the field is uniform and normal to the surface. The charge per unit length on the other surfaces is

$$q' = 2\int_a^\infty \frac{-A_1}{\varepsilon_0}\left(1-\frac{a^2}{r^2}-1\right)dr = \frac{-2A_1}{\varepsilon_0}\frac{a^2}{r}\bigg|_a^\infty = \frac{2A_1 a}{\varepsilon_0},$$

AFTER subtracting off the constant term due to the uniform field which does NOT vanish as $r \to \infty$. This charge precisely balances the extra charge on the half-cylinder. The factor of two in front is because there are two surfaces with the same charge deficit.

4) This problem is a straightforward application of the standard hemisphere problem. First note that if $V/2$ is subtracted from the problem as stated $\Phi'(r,\theta) = \Phi(r,\theta) - V/2$, the problem has four hemispheres at potentials $\pm V/2$, the upper hemisphere of the inner sphere and the lower hemis phere of the outer sphere are positive and the others are now at $-V/2$. The potential may be expanded as

$$\Phi'(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + \frac{B_l}{r^{l+1}}\right]P_l(\cos\theta) \qquad a < r < b$$

The boundary conditions at $r = a, b$, and the work on the hemispheres problem in the text imply

$$\left.\begin{aligned} A_l a^l + \frac{B_l}{a^{l+1}} &= \frac{V}{2}K_l \\ A_l b^l + \frac{B_l}{b^{l+1}} &= -\frac{V}{2}K_l \end{aligned}\right\} \qquad K_l = \left(-\frac{1}{2}\right)^{(l=1)/2}\frac{(2l+1)(l-1)!!}{2\left(\frac{l+1}{2}\right)!}$$

Solving for the expansion coefficients yields

$$A_l = \frac{V}{2}\frac{K_l}{\left[\frac{a^l}{b^{l+1}} - \frac{b^l}{a^{l+1}}\right]}\left(\frac{1}{b^{l+1}} + \frac{1}{a^{l+1}}\right) = \frac{V}{2}\frac{K_l}{\left[a^{2l+1} - b^{2l+1}\right]}\left(a^{l+1} + b^{l+1}\right)$$

$$B_l = -\frac{V}{2}\frac{K_l}{\left[\frac{a^l}{b^{l+1}} - \frac{b^l}{a^{l+1}}\right]}\left(b^l + a^l\right) = -\frac{V}{2}\frac{K_l}{\left[a^{2l+1} - b^{2l+1}\right]}\left(a^{l+1}b^{2l+1} + b^{l+1}a^{2l+1}\right)$$

The potential is

$$\Phi(r,\theta) = \frac{V}{2}\left[1 + \sum_{l=0}^{\infty}\frac{K_l}{\left[a^{2l+1} - b^{2l+1}\right]}\left[\left(a^{l+1} + b^{l+1}\right)r^l - \frac{\left(a^{l+1}b^{2l+1} + b^{l+1}a^{2l+1}\right)}{r^{l+1}}\right]P_l(\cos\theta)\right] \qquad a < r < b,$$

as may be checked by putting $r = a, b$ in the solution. The coefficients of the first four terms are $3/2, 7/8, 11/16, 25/32$. In the limit $b \to \infty$,

$$\Phi(r,\theta) \to \frac{V}{2}\left[1 + \sum_{l=0}^{\infty} K_l \frac{a^{l+1}}{r^{l+1}} P_l(\cos\theta)\right]$$
$$= \frac{V}{2}\left[1 + \sum_{l=0}^{\infty} K_l \frac{a^{l+1}}{r^{l+1}} P_l(\cos\theta)\right],$$

the solution to the two hemisphere problem for $r > a$. In the limit $a \to 0$,

$$\Phi(r,\theta) \to \frac{V}{2}\left[1 - \sum_{l=0}^{\infty} K_l \frac{r^{l}}{b^{l}} P_l(\cos\theta)\right]$$
$$= \frac{V}{2}\left[1 - \sum_{l=0}^{\infty} K_l \frac{r^{l}}{b^{l+1}} P_l(\cos\theta)\right],$$

as it should.