Physics 604
Problem Set 2
Due Oct. 07, 2010

1) a) Let the plane be the xy plane and the charge be on the z -axis. The charge and its image have potential

$$\Phi(x,y,z) = -\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+(z+d)^2}} + \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+(z-d)^2}}.$$

The normal electric field is

$$E_z = -\frac{\partial\Phi}{\partial z}\bigg|_{z=0} = -\frac{q}{4\pi\varepsilon_0}\frac{d}{\left(x^2+y^2+d^2\right)^{3/2}} + \frac{q}{4\pi\varepsilon_0}\frac{-d}{\left(x^2+y^2+d^2\right)^{3/2}}$$

$$= -\frac{q}{4\pi\varepsilon_0}\frac{2d}{\left(x^2+y^2+d^2\right)^{3/2}}.$$

The surface density is

$$\sigma(x,y) = -\frac{q}{4\pi}\frac{2d}{\left(x^2+y^2+d^2\right)^{3/2}}.$$

b) The force between the charge and its image is

$$\vec{F} = -\frac{q^2}{4\pi\varepsilon_0(2d)^2}\hat{z}.$$

c) The total force acting on the plane is

$$F = \frac{\hat{z}q^2}{2\varepsilon_0}\int_0^\infty\int_0^{2\pi}\frac{(2d)^2}{(4\pi)^2\left(x^2+y^2+d^2\right)^3}rdrd\theta$$

$$= \frac{2\pi\hat{z}q^2}{4\varepsilon_0}\int_0^\infty\frac{(2d)^2}{(4\pi)^2\left(X+d^2\right)^3}dX = -\frac{\hat{z}q^2(2d)^2}{32\pi\varepsilon_0}\frac{1}{-2\left(X+d^2\right)^2}\bigg|_0^\infty$$

$$= \frac{\hat{z}q^2}{4\pi\varepsilon_0(2d)^2}.$$

d) The work to remove the particle to infinity is

$$W = -\int_d^\infty \vec{F}\cdot d\vec{l} = \frac{q^2}{16\pi\varepsilon_0}\int_d^\infty \frac{dr}{r^2} = \frac{q^2}{16\pi\varepsilon_0 d}$$

e) The potential energy between the charge and its image is clearly

$$"W" = \frac{q^2}{4\pi\varepsilon_0 2d} = \frac{q^2}{8\pi\varepsilon_0 d}$$

This is not equal to the answer in d) because the conditions are different in the derivation. In d) the image charge is moving as the work is accumulated. The standard derivation in e) assumes that succeeding charges are fixed as they are added to the previous charges. The extra energy needed accounts for the extra factor of two in e).

f) Numerically

$$W = \frac{q^2}{16\pi\varepsilon_0 d} = \frac{\left(1.602\times10^{-19}\ \mathrm{C}\right)^2}{16\pi\left(8.854\times10^{-11}\ \mathrm{C^2/N\,m^2}\right)\left(10^{-10}\ \mathrm{m}\right)} = \frac{1.602\times10^{-9}}{16\pi 8.854\times10^{-11}}\ \mathrm{eV}$$
$$= 3.600\ \mathrm{eV}.$$

2) a) This problem is a straightforward repeat of the derivation in Section 2.2 of Jackson. Let $\vec{x}'$ be the location of the charge $|\vec{x}'| < a$, and therefore $|\vec{x}'|/a < 1$. Let $\vec{x}''$ be the position of the image. By symmetry the image charge lies on the same radius vector as the charge. An argument paralleling that in 2.2 shows

$$\frac{q}{a} = -\frac{q'}{|\vec{x}''|} \qquad\qquad \frac{|\vec{x}'|}{a} = \frac{a}{|\vec{x}''|}$$

The potential inside the sphere is therefore

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{|\vec{x}'|\left|\vec{x} - \frac{a^2}{|\vec{x}'|^2}\vec{x}'\right|}\right] \qquad |\vec{x}| < a$$

b) The electric field may be determined by either differentiation or by Coulomb's law. The result is

$$\vec{E}(\vec{x}) = -\vec{\nabla}\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} - \frac{a\left(\vec{x}-\frac{a^2}{|\vec{x}'|^2}\vec{x}'\right)}{|\vec{x}'|\left|\vec{x}-\frac{a^2}{|\vec{x}'|^2}\vec{x}'\right|^3}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} + \frac{\left(\vec{x}'-\frac{|\vec{x}'|^2}{a^2}\vec{x}\right)}{\left|\frac{|\vec{x}'|}{a}\vec{x}-\frac{a}{|\vec{x}'|}\vec{x}'\right|^3}\right]$$

$$\sigma = \varepsilon_0\vec{E}\left(|\vec{x}|=a\right)\cdot\vec{n} =$$

$$\frac{q}{4\pi}\left[\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} + \frac{\left(\vec{x}'-\frac{|\vec{x}'|^2}{a^2}\vec{x}\right)}{\left|\frac{|\vec{x}'|}{a}\vec{x}-\frac{a}{|\vec{x}'|}\vec{x}'\right|^3}\right]\cdot\left(-\vec{x}/|\vec{x}|\right) =$$

$$-\frac{q}{4\pi}\left[\frac{a-|\vec{x}'|\cos\theta}{\left(a^2+|\vec{x}'|^2-2a|\vec{x}'|\cos\theta\right)^{3/2}} + \frac{\left(|\vec{x}'|\cos\theta-\frac{|\vec{x}'|^2}{a^2}a\right)}{\left(a^2+|\vec{x}'|^2-2a|\vec{x}'|\cos\theta\right)^{3/2}}\right]$$

$$= -\frac{q}{4\pi}\frac{a\left(1-\frac{|\vec{x}'|^2}{a^2}\right)}{\left(a^2+|\vec{x}'|^2-2a|\vec{x}'|\cos\theta\right)^{3/2}} = -\frac{q}{4\pi a^2}\frac{\left(1-\frac{|\vec{x}'|^2}{a^2}\right)}{\left(1+|\vec{x}'|^2/a^2-2\left(|\vec{x}'|/a\right)\cos\theta\right)^{3/2}}$$

c) The force may be obtained by integrating Coulomb's law over this surface charge. It is far easier (and correct!) just to evaluate the force between the particle and its image. The result is

$$|F| = \frac{qq'}{4\pi\varepsilon_0|\vec{x}'-\vec{x}''|^2} = \frac{q^2}{4\pi\varepsilon_0}\frac{a}{|\vec{x}'|}\frac{1}{\left(\frac{a^2}{|\vec{x}'|}-|\vec{x}'|\right)^2} = \frac{q^2}{4\pi\varepsilon_0}\frac{a|\vec{x}'|}{\left(a^2-|\vec{x}'|^2\right)^2}.$$

The force is directed radially outward. Note that in the limit $|\vec{x}'| \to a$ the force goes to that of an charge interacting with a grounded plane (result in problem 1), as it should.

d) If the sphere is held at a potential of V, there is a charge

$$\hat{q} = 4\pi\varepsilon_0 aV$$

residing on the outside surface of the sphere. There is no change of the electric field inside of the sphere. Finally, clearly the total charge on the inside surface of the sphere is $-q$ (the diligent among you will check by integrating σ !) If the total charge on the sphere is Q, by superposition the potential on the sphere is

$$V_{sphere} = \frac{Q-q}{4\pi\varepsilon_0 a},$$

and the electric field is radial and has $E_r = (Q-q)/(4\pi\varepsilon_0 r^2)$ outside the sphere. Inside the sphere the electric field is the same.

3) a) As mentioned in class, use 2 negative image line charges at $\vec{x} = (-x_0, y_0)$ and $\vec{x} = (x_0, -y_0)$ and a positive image line charge at $\vec{x} = (-x_0, -y_0)$ to get the potential for $x, y > 0$ inside the corner. The summed potential, using the usual rules of the natural logarithm is

$$\Phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0} \ln \frac{\left((x-x_0)^2 + (y+y_0)^2\right)\left((x+x_0)^2 + (y-y_0)^2\right)}{\left((x-x_0)^2 + (y-y_0)^2\right)\left((x+x_0)^2 + (y+y_0)^2\right)}.$$

If either $x = 0$ or $y = 0$ the argument of the natural logarithm is 1, and the potential vanishes. Computing the electric field in the x-direction

$$E_x = -\frac{\partial \Phi}{\partial x} = \frac{\lambda}{4\pi\varepsilon_0} \ln \left[\frac{\left((x-x_0)^2 + (y+y_0)^2\right)\left((x+x_0)^2 + (y-y_0)^2\right)}{\left((x-x_0)^2 + (y-y_0)^2\right)\left((x+x_0)^2 + (y+y_0)^2\right)} \right] \frac{\partial}{\partial x}[\ ]$$
$$= 0,$$

Because the argument of the natural logarithm is still l. A similar argument works to show that E_y vanishes on the boundary.

$$C = q/\Delta\Phi = \frac{\varepsilon_0 A}{d}.$$

b) Here assume that the negative charge is on the inside sphere, and uniformly distributed by symmetry. Gauss's Law gives

$$4\pi r^2 E_r = \frac{-q}{\varepsilon_0} \rightarrow E_r = \frac{-q}{4\pi\varepsilon_0} \frac{1}{r^2}.$$

The potential difference is

$$\Delta\Phi = -\int_a^b E_r dr = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right).$$

The capacitance is

$$C = \frac{4\pi\varepsilon_0 ab}{b-a}.$$

c) Assume that the negative charge is on the inside cylinder and uniformly distributed along the length of the cylinder. Gauss's Law gives

$$2\pi r E_r \Delta L = \frac{-(q/L)}{\varepsilon_0}\Delta L \to E_r = \frac{-(q/L)}{2\pi\varepsilon_0}\frac{1}{r}.$$

The potential difference is

$$\Delta\Phi = -\int_a^b E_r dr = \frac{q/L}{2\pi\varepsilon_0}(\ln b - \ln a).$$

The capacitance is

$$C = 2\pi\varepsilon_0 L / \ln(b/a).$$

d) One

4) a) The work needed to remove a charge from a grounded sphere is

$$W = -\int \vec{F}\cdot d\vec{l} = \frac{q^2}{4\pi\varepsilon_0 a^2}\int_r^\infty \frac{a^3}{y^3}\left(1-\frac{a^2}{y^2}\right)^{-2} dy$$

$$= \frac{q^2 a}{4\pi\varepsilon_0}\int_r^\infty \frac{y dy}{\left(y^2-a^2\right)^2}$$

$$\frac{q^2 a}{8\pi\varepsilon_0}\frac{-1}{\left(y^2-a^2\right)}\Bigg|_r^\infty$$

$$= \frac{q^2 a}{8\pi\varepsilon_0}\frac{1}{\left(r^2-a^2\right)}.$$

This result, however is a little different than the usual one. If one evaluates the electrostatic potential of the image acting back on the original charge one obtains a t because the potential itself depends on the location of the charge as it is moving.edge effects and the corrections due to the fact the charge is slightly displaced from uniform distribution on the conductor (corrections of order a/d and b/d), and assume that the fields are simply two line charge fields superposed.

b) This part follow from the fact that the second term in the force formula is (unbelievably!) a perfect differential

$$W = -\int \vec{F}\cdot d\vec{l} = -\frac{q}{4\pi\varepsilon_0}\int_r^\infty \left[\frac{Q}{y^2} - \frac{qa^3\left(2y^2-a^2\right)}{y^3\left(y^2-a^2\right)^2}\right]dy$$

$$= -\frac{qQ}{4\pi\varepsilon_0 r} + \frac{q}{4\pi\varepsilon_0}\int_r^\infty \frac{qa^3\left(2y^3-a^2y\right)}{\left(y^4-a^2y^2\right)^2}dy$$

$$= -\frac{qQ}{4\pi\varepsilon_0 r} - \frac{q^2a^3}{8\pi\varepsilon_0}\frac{1}{\left(y^4-a^2y^2\right)}\Bigg|_r^\infty$$

$$= -\frac{qQ}{4\pi\varepsilon_0 r} + \frac{q^2a^3}{8\pi\varepsilon_0}\frac{1}{\left(r^4-a^2r^2\right)} = -\frac{qQ}{4\pi\varepsilon_0 r} + \frac{q^2a}{8\pi\varepsilon_0}\left[\frac{1}{r^2-a^2} - \frac{1}{r^2}\right].$$

It should be noted that a posted solution many people used seems incorrect. The integral is not done properly! Note that the third term is from the fact that the insulated sphere has a potential $V_{sphere} = \frac{q^2}{4\pi\varepsilon_0 r}$ which changes as r does. The total electric field goes goes as 1/r3 because of this effect.

where $\vec{r}_{a,b}$ are the positions of the center of the line charges. To the first significant order in a/d and b/d,

$$\Delta\Phi\left(\vec{x}\right) \approx \frac{q/L}{2\pi\varepsilon_0}\ln\left|d/a\right| - \frac{q/L}{2\pi\varepsilon_0}\ln\left|d/b\right|$$

$$= \frac{q/L}{2\pi\varepsilon_0}\ln\left|\frac{d^2}{ab}\right| = \frac{q/L}{\pi\varepsilon_0}\ln\left|\frac{d}{\sqrt{ab}}\right|$$

5) This problem is a straightforward application of the method of images and the surface integral form of the solution Eqn. 1.44.

a) The method of images gives the Green function. If $\vec{x} = \left(x, y, z\right)$ is the observation point and $\vec{x}' = \left(x', y', z'\right)$ is the source point, then

$$G_D(\vec{x},\vec{x}') = \frac{1}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)} - \frac{1}{\left((x-x')^2+(y-y')^2+(z+z')^2\right)}$$

automatically gives the Dirichtlet condition on the surface $z'=0$.

b) Using

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')d^3x' - \frac{1}{4\pi}\int_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}da',$$

the fact $\rho(\vec{x})=0$ for the Laplace Equation, and $\partial G/\partial n' = -\partial G/\partial z'$,

$$\Phi(\vec{x}) = \frac{1}{4\pi}\int_S \Phi(\vec{x}')\left[\begin{array}{l}\dfrac{z-z'}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{3/2}} \\ +\dfrac{z+z'}{\left((x-x')^2+(y-y')^2+(z+z')^2\right)^{3/2}}\end{array}\right]da'$$

$$= \frac{V}{2\pi}\int_0^a\int_0^{2\pi}\left[\frac{z}{\left(\rho^2+2\rho r'\cos(\theta-\theta')+r'^2+z^2\right)^{3/2}}\right]r'dr'd\theta',$$

when expressed in terms of cylindrical coordinates.

c) When $\rho=0$ the integral follows from elementary integration theory

$$\Phi(\vec{x}) = \frac{V}{2\pi}\int_0^a\int_0^{2\pi}\left[\frac{z}{\left(r'^2+z^2\right)^{3/2}}\right]r'dr'd\theta'$$

$$= Vz\int_0^a \frac{r'dr'}{\left(r'^2+z^2\right)^{3/2}} = \frac{Vz}{2}\int_0^{a^2}\frac{dx}{\left(x+z^2\right)^{3/2}}$$

$$= -Vz\int_0^{a^2} d\left(x+z^2\right)^{-1/2} = -Vz\left[\left(a^2+z^2\right)^{-1/2} - z^{-1}\right]$$

$$= V\left[1-\frac{z}{\sqrt{a^2+z^2}}\right]$$

d) Expanding by the Binomial TheoremGreen's Second Identity. Following the same procedure a

$$
\begin{aligned}
\Phi(\vec{x}) &= \frac{1}{4\pi}\int_S \left(\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right) da' \\
&= \frac{1}{4\pi a}\int_S \vec{\nabla}\Phi\cdot\vec{n}' da' + \frac{1}{4\pi}\int_S \frac{\Phi}{R^2} da' \\
&= \frac{1}{4\pi a}\int_V \nabla^2\Phi dx'dy'dz' + \frac{1}{4\pi a^2}\int_S \Phi da' \\
&= \frac{1}{4\pi a^2}\int_S \Phi(\vec{x}') da',
\end{aligned}
$$

where a is the radius of the sphere and $R = |\vec{x}-\vec{x}'|$. The final integral is clearly the average of the potential over the surface of the sphere. It does not matter what radius is chosen for the sphere in performing the average, but of course the values of the scalar potential on the surface *will* depend on the choice of radius.

6) The (upper bound) capacitance determined by the trial function is

$$
\begin{aligned}
C[\psi] &= \varepsilon_0 \int_V \left|\vec{\nabla}\psi\right|^2 d^3\vec{x} \\
&= \varepsilon_0 \frac{2\pi L}{(b-a)^2}\int_a^b r dr \\
&= \varepsilon_0 \frac{2\pi L}{(b-a)^2}\left[\frac{b^2}{2} - \frac{a^2}{2}\right] \\
&= \varepsilon_0 \pi L \frac{b+a}{b-a}
\end{aligned}
$$

where a is the inner radius of the cylinder, where b is the outer radius of the cylinder, and L is the length of the cylinder. Evaluating the exact and estimated capacitance numerically yields this table:

b/a	Exact $[C/2\pi\varepsilon_0 L] = (\ln b/a)^{-1}$	Trial Function $[C/2\pi\varepsilon_0 L] = (b/a+1)/(2b/a-2)$
1.5	2.46630	2.5
2	1.44270	1.5
3	0.91024	1.0

The "Exact" field is more like the linear trial function when $b \to a$. In the limit, clearly the two expressions agree by the expansion $\ln(1+x) \to x$ for small x. Notice the trial values are indeed higher than the exact values.