Physics 417/517
Introduction to Particle Accelerator Physics

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Comment on Strong Focusing

Last time neglected to mention one main advantage of strong focusing. In weak focusing machines, $n < 1$ for stability. Therefore, the fall-off distance, or field gradient cannot be too high. There is no such limit for strong focusing.

$$n \gg 1$$

is now allowed, leading to large field gradients and relatively short focal length magnetic lenses. This tighter focusing is what allows smaller beam sizes. Focusing gradients now limited only by magnet construction issues (pole magnetic field limits).

Linear Beam Optics Outline

- Particle Motion in the Linear Approximation
- Some Geometry of Ellipses
- Ellipse Dimensions in the β-function Description
- Area Theorem for Linear Transformations
- Phase Advance for a Unimodular Matrix
  - Formula for Phase Advance
  - Matrix Twiss Representation
  - Invariant Ellipses Generated by a Unimodular Linear Transformation
- Detailed Solution of Hill's Equation
  - General Formula for Phase Advance
  - Transfer Matrix in Terms of β-function
  - Periodic Solutions
- Non-periodic Solutions
  - Formulas for β-function and Phase Advance
- Beam Matching

Linear Particle Motion

Fundamental Notion: The *Design Orbit* is a path in an Earth-fixed reference frame, i.e., a differentiable mapping from [0,1] to points within the frame. As we shall see as we go on, it generally consists of *arcs of circles* and *straight lines*.

$$\sigma : [0,1] \to \mathrm{R}^3$$

$$\sigma \quad \to \vec{X}(\sigma) = \left(X(\sigma), Y(\sigma), Z(\sigma) \right)$$

Fundamental Notion: *Path Length*

$$ds = \sqrt{\left(\frac{dX}{d\sigma}\right)^2 + \left(\frac{dY}{d\sigma}\right)^2 + \left(\frac{dZ}{d\sigma}\right)^2}\, d\sigma$$

The *Design Trajectory* is the path specified in terms of the path length in the Earth-fixed reference frame. For a relativistic accelerator where the particles move at the velocity of light, $L_{tot}=ct_{tot}$.

$$s:[0,L_{tot}]\rightarrow \mathrm{R}^3$$

$$s \;\rightarrow\; \vec{X}(s)=\left(X(s),Y(s),Z(s)\right)$$

The first step in designing any accelerator, is to specify bending magnet locations that are consistent with the arc portions of the Design Trajectory.

Betatron Design Trajectory

$$s:[0,2\pi R]\rightarrow \mathrm{R}^3$$

$$s \;\rightarrow\; \vec{X}\left(s\right)=\left(R\cos\left(s/R\right),R\sin\left(s/R\right),0\right)$$

Use path length s as independent variable instead of t in the dynamical equations.

$$\frac{d}{ds}=\frac{1}{\Omega_c R}\frac{d}{dt}$$

Betatron Motion in *s*

$$\frac{d^2\delta r}{dt^2}+(1-n)\Omega_c^2\delta r=\Omega_c^2 R\frac{\Delta p}{p}$$

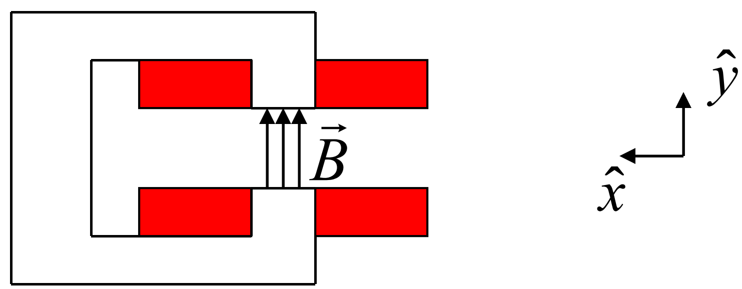
$$\frac{d^2\delta z}{dt^2}+n\Omega_c^2\delta z=0$$

$$\Downarrow$$

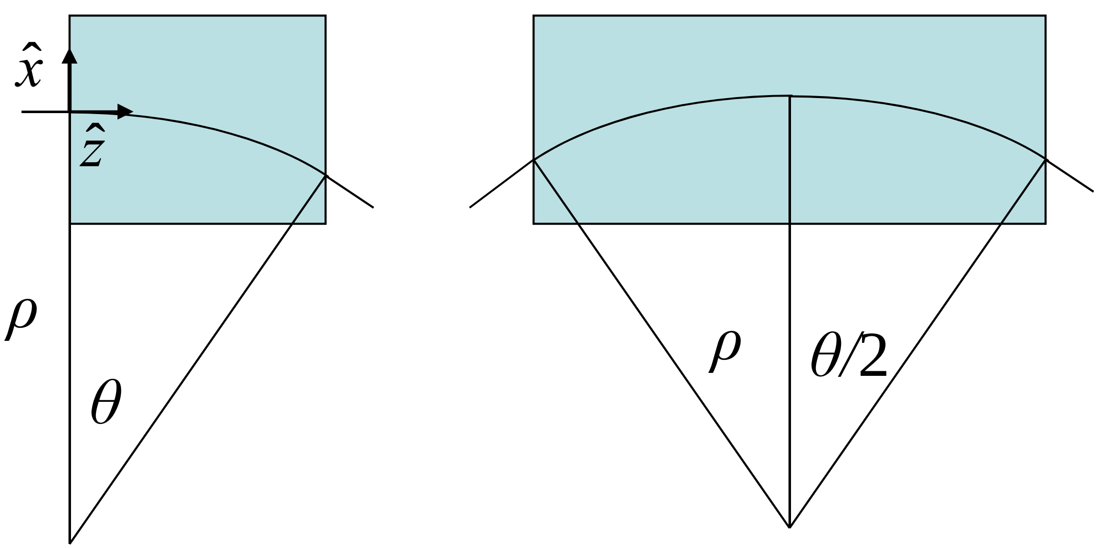
$$\frac{d^2\delta r}{ds^2}+\frac{(1-n)}{R^2}\delta r=\frac{1}{R}\frac{\Delta p}{p}$$

$$\frac{d^2\delta z}{ds^2}+\frac{n}{R^2}\delta z=0$$

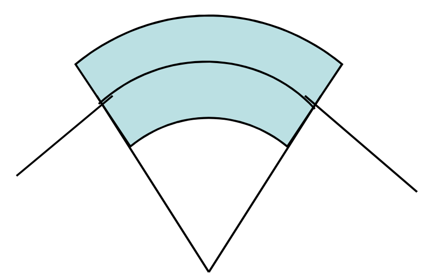
Bend Magnet Geometry



Rectangular Magnet of Length L



Sector Magnet



Bend Magnet Trajectory

For a uniform magnetic field

$$\frac{d\left(\gamma m\vec{V}\right)}{dt}=\left[\vec{E}+\vec{V}\times\vec{B}\right]$$

$$\frac{d\left(\gamma m V_x\right)}{dt}=-qV_zB_y$$

$$\frac{d\left(\gamma m V_z\right)}{dt}=qV_xB_y$$

$$\frac{d^2V_x}{dt^2}+\Omega_c^2V_x=0 \qquad \frac{d^2V_z}{dt^2}+\Omega_c^2V_z=0$$

For the solution satisfying boundary conditions: $\vec{X}\left(0\right)=0$ $\quad \vec{V}\left(0\right)=V_{0z}\hat{z}$

$$X\left(t\right)=\frac{p}{qB_y}\left(\cos\left(\Omega_c t\right)-1\right)=\rho\left(\cos\left(\Omega_c t\right)-1\right) \qquad \Omega_c=qB_y/\gamma m$$

$$Z\left(t\right)=\frac{p}{qB_y}\sin\left(\Omega_c t\right)=\rho\sin\left(\Omega_c t\right)$$

Magnetic Rigidity

The magnetic rigidity is:

$$B\rho = \left|B_y \rho\right| = \frac{p}{|q|}$$

It depends only on the particle momentum and charge, and is a convenient way to characterize the magnetic field. Given magnetic rigidity and the required bend radius, the required bend field is a simple ratio. Note particles of momentum 100 MeV/c have a rigidity of 0.334 T m.

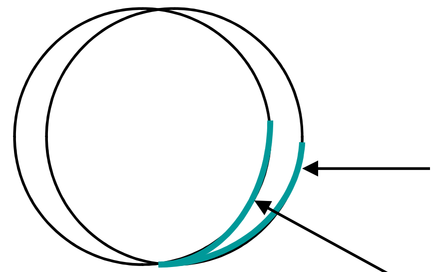
Long Dipole Magnet

$$BL = B\rho\left(2\sin\left(\theta/2\right)\right)$$

Normal Incidence (or exit) Dipole Magnet

$$BL = B\rho\sin\left(\theta\right)$$

Natural Focusing in Bend Plane



Can show that for either a displacement perturbation or angular perturbation from the design trajectory

$$\frac{d^2x}{ds^2} = -\frac{x}{\rho_x^2(s)} \qquad \frac{d^2y}{ds^2} = -\frac{y}{\rho_y^2(s)}$$

Quadrupole Focusing

$$\vec{B}(x,y) = B'(s)(x\hat{y} + y\hat{x})$$

$$\gamma m \frac{d\mathrm{v}_{\mathrm{x}}}{ds} = -qB'(s)x \qquad \gamma m \frac{d\mathrm{v}_{\mathrm{y}}}{ds} = qB'(s)y$$

$$\frac{d^2x}{ds^2} + \frac{B'(s)}{B\rho}x = 0 \qquad \frac{d^2y}{ds^2} - \frac{B'(s)}{B\rho}y = 0$$

Combining with the previous slide

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho}\right]x = 0 \qquad \frac{d^2y}{ds^2} + \left[\frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}\right]y = 0$$

Hill's Equation

Define focusing strengths (with units of m^{-2})

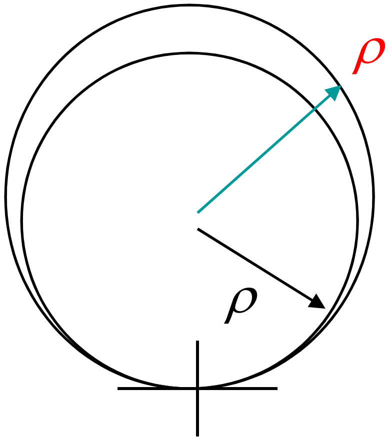
$$k_x(s) = \frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho} \qquad k_y = \frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}$$

$$\frac{d^2x}{ds^2} + k_x(s)x = 0 \qquad \frac{d^2y}{ds^2} + k_y(s)y = 0$$

Note that this is like the harmonic oscillator, or exponential for constant *K,* but more general in that the focusing strength, and hence oscillation frequency depends on *s*

Energy Effects

$$\Delta x(s) = \frac{p}{eB_y}\frac{\Delta p}{p}\left(1-\cos\left(s/\rho\right)\right)$$



This solution is not a solution to Hill's equation directly, but *is* a solution to the inhomogeneous Hill's Equations

$$\frac{d^2x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]x = \frac{1}{\rho_x(s)}\frac{\Delta p}{p}$$

$$\frac{d^2y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]y = \frac{1}{\rho_y(s)}\frac{\Delta p}{p}$$