Physics 417/517 Introduction to Particle Accelerator Physics

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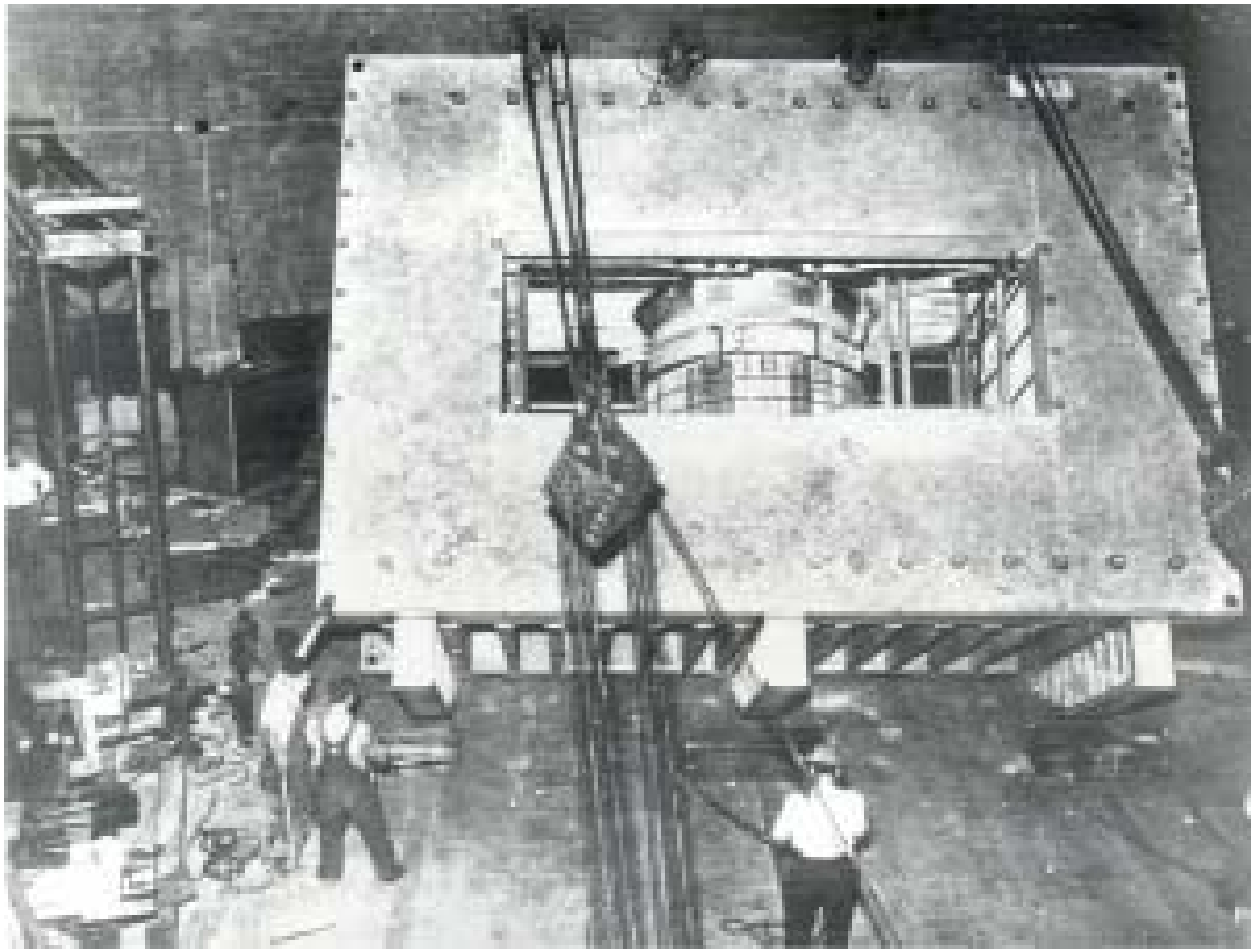
Betatrons

25 MeV electron accelerator with its inventor: Don Kerst. The earliest electron accelerators for medical uses were betatrons.

300 MeV ~ 1949



Electromagnetic Induction

Faraday's Law: Differential Form of Maxwell Equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

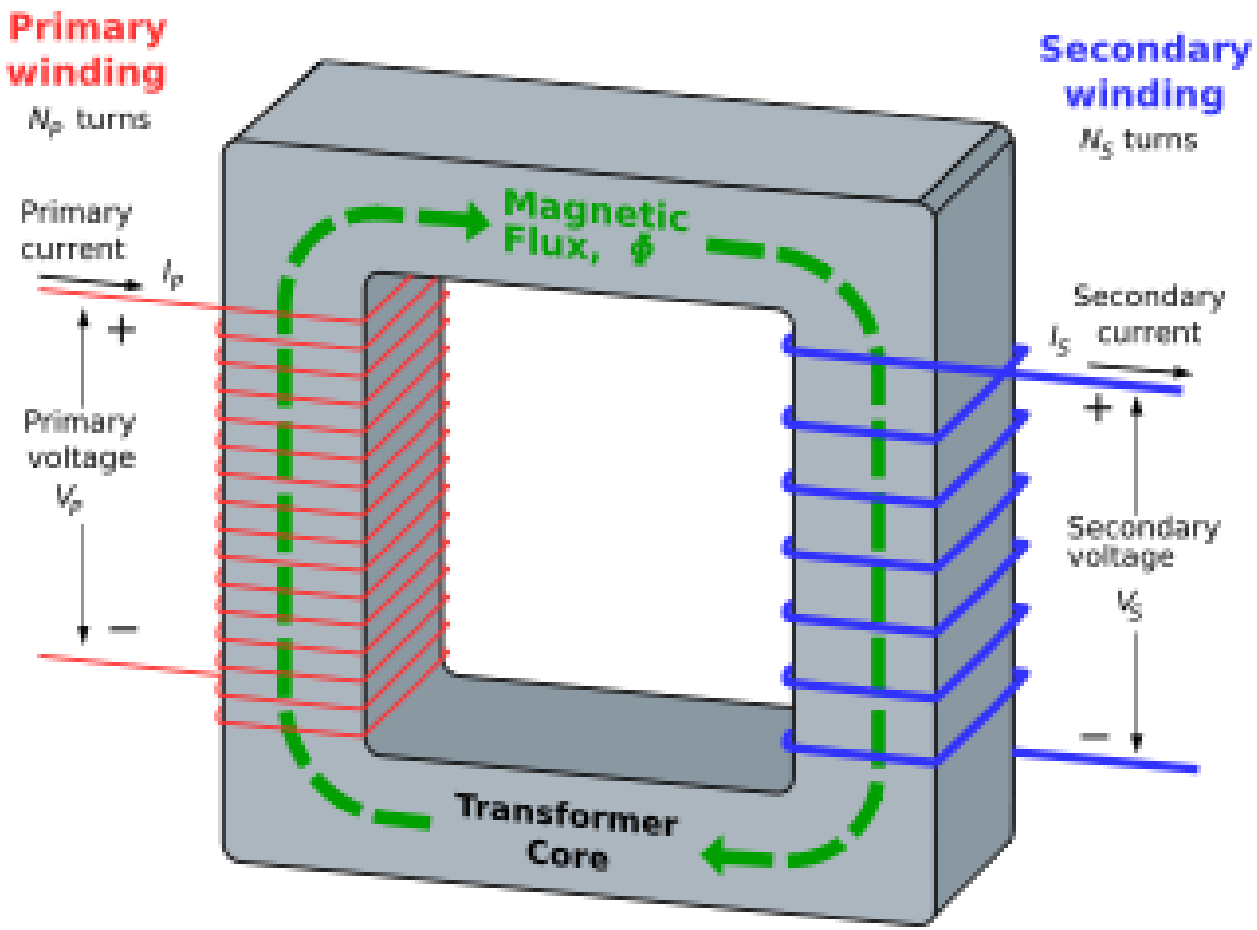
Faraday's Law: Integral Form

$$\oint_S \vec{\nabla} \times \vec{E} \cdot d\vec{S} = -\oint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

Faraday's Law of Induction

$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = 2\pi R E_\theta = -\frac{d}{dt}\Phi_B$$

Transformer



Betatron as a Transformer

- In the betatron the electron beam itself is the secondary winding of the transformer. Energy transferred directly to the electrons

$$2\pi R E_\theta = -\frac{d}{dt}\Phi_B$$

- Radial Equilibrium

$$R = \frac{\beta c}{eB/\gamma m}$$

- Energy Gain Equation

$$\frac{d\gamma}{dt} = \frac{eE_\theta \beta c}{mc^2}$$

Betatron condition

To get radial stability in the electron beam orbit (i.e., the orbit radius does not change during acceleration), need

$$R = const \Rightarrow \frac{dB}{dt} = \frac{B}{\gamma}\frac{d\gamma}{dt} \text{ and } \frac{B}{\gamma} \approx \frac{cm}{eR}$$

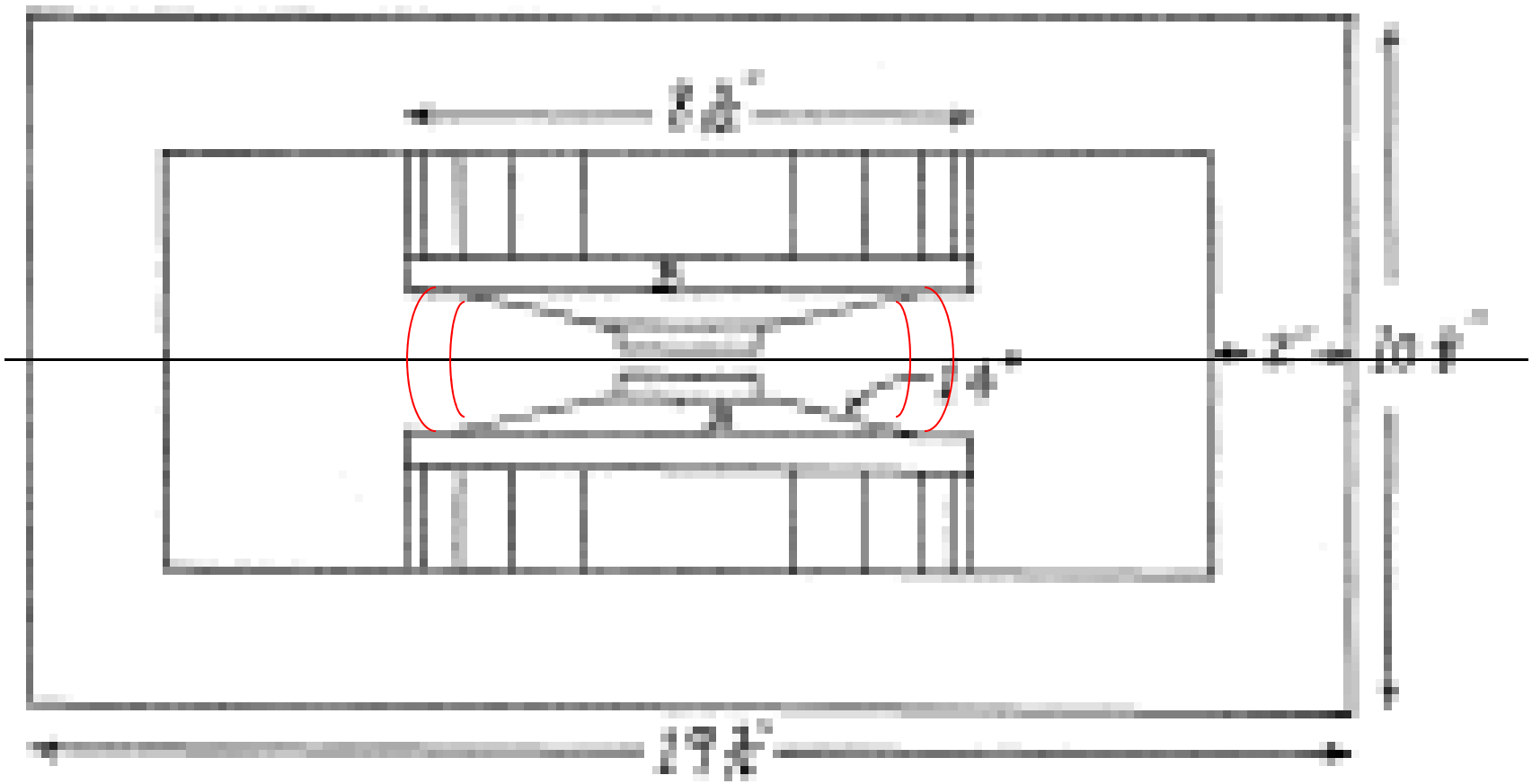
$$\Phi_B = \alpha \pi R^2 B \text{ for some } \alpha \text{ and } \frac{d\gamma}{dt} \approx \frac{ec}{mc^2}\frac{1}{2\pi R}\frac{d\Phi_B}{dt} \Rightarrow \alpha = 2$$

$$\therefore \Phi_B = 2\pi R^2 B\left(r = R\right)$$

This last expression is sometimes called the "betatron two for one" condition. The energy increase from the flux change is

$$\gamma - \gamma_0 \approx \frac{q\beta c}{2\pi R mc^2}\Delta\Phi_B$$

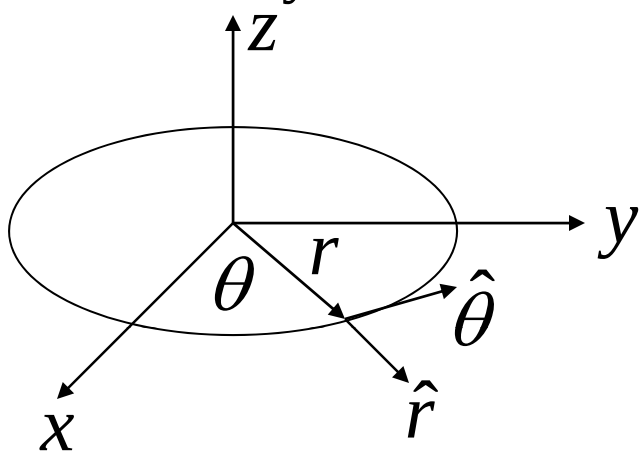
Transverse Beam Stability



Ensured by proper shaping of the magnetic field in the betatron

Relativistic Equations of Motion

Standard Cylindrical Coordinates



$$\frac{d\vec{\mathrm{v}}}{dt} = \frac{q}{\gamma m}\left(\vec{\mathrm{v}} \times \vec{B}\right) \qquad \frac{d\gamma}{dt} = 0!!$$

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \qquad \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\mathrm{v}_r = \vec{\mathrm{v}} \cdot \hat{r} = \dot{r} \qquad \mathrm{v}_\theta = \vec{\mathrm{v}} \cdot \hat{\theta} = r\dot{\theta}$$

$$\frac{d\vec{\mathrm{v}}}{dt} = \frac{d}{dt}\left(\mathrm{v}_r \hat{r} + \mathrm{v}_\theta \hat{\theta}\right)$$

$$d\hat{r}/dt = \dot{\theta}\hat{\theta}$$

$$d\hat{\theta}/dt = -\dot{\theta}\hat{r}$$

Cylindrical Equations of Motion

In components

$$\ddot{r} - r\dot{\theta}^2 = \frac{q}{\gamma m}\left(\vec{\mathrm{v}} \times \vec{B}\right)_r = \frac{q}{\gamma m} r\dot{\theta} B_z$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{q}{\gamma m}\left(\vec{\mathrm{v}} \times \vec{B}\right)_\theta = \frac{q}{\gamma m}\left(\dot{z}B_r - \dot{r}B_z\right)$$

$$\ddot{z} = \frac{q}{\gamma m}\left(\vec{\mathrm{v}} \times \vec{B}\right)_z = -\frac{q}{\gamma m} r\dot{\theta} B_r$$

Zero'th order solution

$$r(t) = cons = R$$

$$\theta(t) = \theta_0 + \dot{\theta}_0 t \qquad z(t) = 0$$