Physics 417/517 Introduction to Particle Accelerator Physics

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$$\frac{d^2 \Delta E}{dt^2} + \frac{1}{T_0} \frac{dW}{dE} \frac{\Delta E}{dt} + \omega_s^2 \Delta E = 0$$

$$\Delta E\left(t\right) \propto e^{i\omega t} \rightarrow \omega_s^2 + 2\alpha_s i\omega - \omega^2 = 0$$

$$\omega = \alpha_s i \pm \sqrt{\omega_s^2 - \alpha_s^2} \approx \pm\omega_s + i\alpha_s$$

Normal modes are damped with a damping given by

$$\alpha_s = \frac{1}{2T_0} \frac{dW}{dE}$$

Power in terms of Local *B* Field

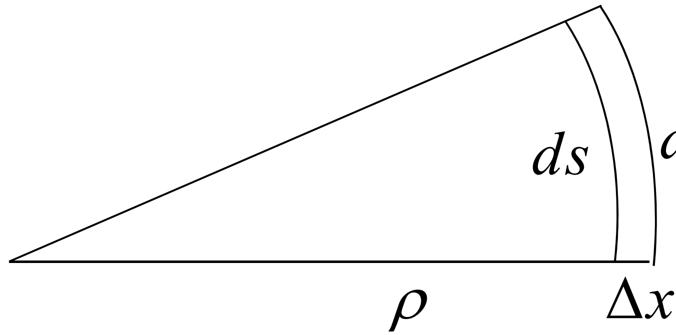
$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^4} \frac{E^4}{\rho^2} = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^4} \frac{E^4 B^2}{\left(B\rho\right)^2}$$

$$= \frac{e^4 c^3}{6\pi\varepsilon_0 \left(m_0 c^2\right)^4} E^2 B^2$$

$$\equiv C E^2 B^2$$

Significance: Given orbit, only need to know the magnetic field on orbit to evaluate total energy emitted

Partition Knowing Linear Optics



$$ds' = \left(1 + \frac{\Delta x}{\rho}\right) ds$$

$$W = \int P dt = \int P_s \frac{ds'}{c} = \frac{1}{c}\int P_s \left(1 + \frac{\Delta x}{\rho}\right) ds = \frac{1}{c}\int P_s \left(1 + \frac{D}{\rho}\frac{\Delta E}{E}\right) ds$$

$$\frac{dW}{dE} = \frac{1}{c}\int \left[\frac{dP_s}{dE} + \frac{D}{\rho}\frac{P_s}{E}\right] ds$$

$$\frac{dP_s}{dE} = 2CEB^2 + 2CE^2 B \frac{dB}{dE}$$

$$= 2\left(\frac{CE^2 B^2}{E} + \frac{CE^2 B^2}{B} \frac{dB}{dx} \frac{dx}{dE} \right)$$

$$= 2P_s \left(\frac{1}{E} + \frac{1}{B} \frac{dB}{dx} \frac{D}{E} \right)$$

$$\frac{dW}{dE} = \frac{1}{c} \int \left[2P_s \left(\frac{1}{E} + \frac{1}{B} \frac{dB}{dx} \frac{D}{E} \right) + P_s \frac{D}{\rho E} \right] ds$$

$$= \frac{2W_0}{E} + \frac{1}{cE} \int DP_s \left[\frac{2}{B} \frac{dB}{dx} + \frac{1}{\rho} \right] ds$$

$$\alpha_s = \frac{1}{2T_0}\frac{dW}{dE} = \frac{1}{c}\int\left[2P_s\left(\frac{1}{E}+\frac{1}{B}\frac{dB}{dx}\frac{D}{E}\right)+P_s\frac{D}{\rho E}\right]ds$$

$$= \frac{W_0}{2T_0 E}\left(2+I\right)$$

$$I = \frac{1}{W_0 c}\int DP_s\left[\frac{2}{B}\frac{dB}{dx}+\frac{1}{\rho}\right]ds = \frac{CE^4}{W_0 e^2 c^3}\int\frac{D}{\rho^2}\left[\frac{2}{B}\frac{dB}{dx}+\frac{1}{\rho}\right]ds$$

Replace *dB/dx* with *k* (strictly OK for separated function magnets only (why?)) and evaluate W_0

$$W_0 = \int P_s dt = \frac{CE^4}{e^2c^3}\int\frac{ds}{\rho^2}$$

Radiation Partition

The damping is partitioned into the various degrees of freedom by

$$\vartheta = \frac{\oint \left(D / \rho^3 \right) \left(1 + 2\rho^2 K \right) ds}{\oint ds / \rho^2}$$

By suitable design (i.e., choice of *D, ρ, K*) one can shift greater or less damping between the horizontal and longitudinal degrees of freedom. "Natural" partition has $\vartheta \approx 1$

Equilibrium Energy Spread

Quantized photon emission events act to stimulate motion in all three degrees of freedom. Thus, the oscillations do not damp to zero.

$$\left\langle \Delta A^2 \right\rangle = \left\langle A_1^2 - A_0^2 \right\rangle = \Delta e^2 \qquad \Delta e = \hbar \omega$$

$$\left\langle \frac{dA^2}{dt} \right\rangle = \int_0^\infty \Delta e^2 \frac{d\dot{n}}{d\left(\Delta e\right)} d\Delta e = \dot{N}_{ph} \left\langle \Delta e^2 \right\rangle$$

In equilibrium.

$$\left\langle A^2 \right\rangle = \frac{\tau_{\Delta\phi}}{2} \dot{N}_{ph} \left\langle \Delta e^2 \right\rangle$$

$$\frac{\sigma_E^2}{E^2} = \frac{55}{32\sqrt{3}} \frac{\hbar c}{mc^2} \frac{\gamma^2}{2+\vartheta} \frac{\left\langle 1/\rho^3 \right\rangle}{\left\langle 1/\rho^2 \right\rangle}$$

Equilibrium Emittance

$$\left\langle \delta a^2 \right\rangle = \frac{\Delta e^2}{E_o^2} H\left(s\right)$$

$$H\left(s\right) = \beta D'^2 + 2\alpha DD' + \gamma D^2$$

In equilibrium, averaged over the ring.

$$\left\langle a^2 \right\rangle = \frac{\tau_x}{2} \dot{N}_{ph} \left\langle \Delta e^2 \right\rangle$$

$$\varepsilon_x = \frac{\sigma_x^2}{\beta_x} = \frac{55}{32\sqrt{3}} \frac{\hbar c}{mc^2} \frac{\gamma^2}{1-\vartheta} \frac{\left\langle H / \rho^3 \right\rangle}{\left\langle 1/\rho^2 \right\rangle}$$

Emittance and energy spread increases

For recirculated linacs, there is no equilibrium and similar estimates are used to compute emittance and energy spread increases (for a bend of π = 180°)

$$\Delta\varepsilon_{x,y} = \frac{1}{2cE_0^2}\int \dot{N}_{ph}\left(e^2\right)H\left(s\right)ds = \frac{55C_\gamma \hbar c\left(mc^2\right)^2}{64\pi\sqrt{3}}\gamma^5\int\frac{H}{\rho^3}ds$$

$$\Delta\frac{\sigma_E^2}{E^2} = \frac{5\pi}{32\sqrt{3}}\frac{\hbar c}{mc^2}\frac{\gamma^2}{2+\vartheta}\left(11-\frac{64}{25}\right)\frac{\left\langle 1/\rho^3\right\rangle}{\left\langle 1/\rho^2\right\rangle}$$

$$= \frac{5\pi r_e}{12\sqrt{3}}\frac{\hbar c}{mc^2}\frac{\gamma^5}{2+\vartheta}\left(11-\frac{64}{25}\right)\frac{1}{\rho^2}$$