



Physics 417/517

Introduction to Particle Accelerator Physics

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Spectral Distribution in Beam Frame

Begin with power distribution in beam frame: dipole radiation pattern (single harmonic only when $K \ll 1$; replace γ^* by γ , β^* by β)

$$\frac{dP^*}{d\Omega^*} = \frac{e^2 c}{32\pi^2 \epsilon_0} k^{*4} a^2 \sin^2 \Theta^*$$

Number distribution in terms of wave number

$$\frac{dN_\gamma}{d\Omega^*} = \frac{\alpha}{4} N K^2 \frac{k_y^{*2} + k_z^{*2}}{k^{*2}}$$

Solid angle transformation

$$d\Omega^* = \frac{d\Omega}{\gamma^2 (1 - \beta \cos \theta)^2}$$

Number distribution in beam frame

$$\frac{dN_\gamma}{d\Omega} = \frac{\alpha}{4} NK^2 \frac{\sin^2 \theta \sin^2 \varphi + \gamma^2 (\cos \theta - \beta)^2}{\gamma^4 (1 - \beta \cos \theta)^4}$$

Energy is simply

$$E(\theta) = \hbar \frac{2\pi\beta c}{\lambda_{ID}(1 - \beta \cos \theta)} \quad \hat{E}(\theta) = \frac{1}{(1 - \beta \cos \theta)}$$

Number distribution as a function of normalized lab-frame energy

$$\frac{dN_\gamma}{d\hat{E}} = \frac{\alpha\pi}{4\gamma^2\beta^3} NK^2 \left[\left(\frac{\hat{E}}{\gamma^2} - 1 \right)^2 + \beta^2 \right]$$

Average Energy



Limits of integration

$$\cos \theta = 1 \quad \hat{E} = \frac{1}{1 - \beta} \quad \cos \theta = -1 \quad \hat{E} = \frac{1}{1 + \beta}$$

Average energy is also analytically calculable

$$\langle E \rangle = \frac{\int_0^{\infty} E \frac{dN_{\gamma}}{d\hat{E}} d\hat{E}}{\int_0^{\infty} \frac{dN_{\gamma}}{d\hat{E}} d\hat{E}} = \gamma^2 \hbar 2\pi \beta c / \lambda_{ID} \approx \frac{E_{\max}}{2}$$

Beam *rms* Emittance



Treat the distribution of particles within the beam statistically.
Define single particle distribution function

$$\rho(x, x')$$

And statistical averaging

$$\langle q \rangle \equiv \int q(x, x') \rho(x, x') dx dx'$$

Closest *rms* Fit Ellipses



For zero-centered distributions, i.e., distributions that have zero average value for x and x'

$$\varepsilon_{rms} \equiv \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

$$\beta = \frac{\langle x^2 \rangle}{\varepsilon_{rms}} = \frac{\sigma_x^2}{\varepsilon_{rms}}$$

$$\alpha = -\frac{\langle xx' \rangle}{\varepsilon_{rms}}$$

$$\gamma = \frac{\langle x'^2 \rangle}{\varepsilon_{rms}} = \frac{\sigma_{x'}^2}{\varepsilon_{rms}}$$

Case: Uniformly Filled Ellipse



$$\rho(x, x') = \frac{1}{\pi\varepsilon} \Theta\left(1 - \frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{\varepsilon}\right)$$

Θ here is the Heavyside step function, 1 for positive values of its argument and zero for negative values for its argument

$$\sigma_x^2 = \langle x^2 \rangle = \frac{\varepsilon\beta}{4}$$

$$\langle xx' \rangle = -\frac{\varepsilon\alpha}{4}$$

$$\sigma_{x'}^2 = \langle x'^2 \rangle = \frac{\varepsilon}{4\beta} (1 + \alpha^2)$$

$$\therefore \varepsilon_{rms} = \frac{\varepsilon}{4}$$