Physics 417/517 Introduction to Particle Accelerator Physics

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

Larmor's Formula

For a particle executing non-relativistic motion, the total power emitted in electromagnetic radiation is (Larmor)

$$P(t) = \frac{1}{6\pi\varepsilon_0}\frac{q^2}{c^3}\left|\vec{a}\right|^2 = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{m^2c^3}\left|\dot{\vec{p}}\right|^2$$

Lienard's relativistic generalization: Note both *dE* and *dt* are the fourth component of relativistic 4-vectors when one is dealing with photon emission. Therefore, their ratio must be an Lorentz invariant. The invariant that reduces to Larmor's formula in the non-relativistic limit is

$$P = -\frac{e^2}{6\pi\varepsilon_0 c}\frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau}$$

Insertion Devices

Often periodic magnetic field magnets are placed in beam path of high energy storage rings. The radiation generated by electrons passing through such insertion devices has unique properties.

Field of the insertion device magnet

$$\vec{B}(x,y,z) = B(z)\,\hat{y} \qquad B(z) \approx B_0 \cos\left(2\pi z / \lambda_{ID}\right)$$

Vector potential for magnet (1 dimensional approximation)

$$\vec{A}(x,y,z) = A(z)\,\hat{x} \qquad A(z) \approx \frac{B_0 \lambda_{ID}}{2\pi} \sin\left(2\pi z / \lambda_{ID}\right)$$

Electron Orbit

Uniformity in x-direction means that canonical momentum in the x-direction is conserved

$$\mathrm{v}_x(z) = \frac{eA(z)}{\gamma m} = \frac{K}{\gamma} c \sin\left(2\pi z / \lambda_{ID}\right)$$

$$x(z) = \int \frac{\mathrm{v}_x}{\mathrm{v}_z} dz \approx -\frac{1}{\langle \beta_z \rangle} \frac{K}{\gamma} \frac{\lambda_{ID}}{2\pi} \cos\left(2\pi z / \lambda_{ID}\right)$$

Field Strength Parameter

$$K \equiv \frac{eB_0 \lambda_{ID}}{2\pi mc}$$

Average Velocity

Energy conservation gives that γ is a constant of the motion

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2(z)}$$

Average longitudinal velocity in the insertion device is

$$\beta^{*2} = \langle \beta_z \rangle^2 = 1 - \frac{1}{\gamma^2} - \frac{K^2}{2\gamma^2}$$

Average rest frame has

$$\gamma^{*2} = \frac{1}{1 - \beta^{*2}} = \frac{\gamma^2}{1 + K^2/2}$$

Relativistic Kinematics

In average rest frame the insertion device is Lorentz contracted, and so its wavelength is

$$\lambda^* = \lambda_{ID} / \beta^* \gamma^*$$

The sinusoidal wiggling motion emits with angular frequency

$$\omega^* = 2\pi c / \lambda^*$$

Lorentz transformation formulas for the wave vector

$$k^* = \gamma^* k\left(1 - \beta^* \cos\theta\right)$$
$$k_x^* = k_x = k \sin\theta \cos\varphi$$
$$k_y^* = k_y = k \sin\theta \sin\varphi$$
$$k_z^* = \gamma^* k\left(\cos\theta - \beta^*\right)$$

Insertion Device (FEL) Resonance Condition

Angle transforms as

$$\cos\theta^* = \frac{k_z^*}{k^*} = \frac{\left(\cos\theta - \beta^*\right)}{\left(1 - \beta^* \cos\theta\right)}$$

Wave vector in lab frame has

$$k = \frac{k^*}{\gamma^*\left(1 - \beta^* \cos\theta\right)} = \frac{2\pi\beta^* c}{\lambda_{ID}\left(1 - \beta^* \cos\theta\right)}$$

In the forward direction cos $\theta = 1$

$$\lambda_e \approx \frac{\lambda_{ID}}{2\gamma^{*2}} = \frac{\lambda_{ID}}{2\gamma^2}\left(1 + K^2/2\right)$$

Power Emitted Lab Frame

Larmor/Lienard calculation in the lab frame yields

$$\langle P \rangle = \frac{e^2}{6\pi\varepsilon_0}\gamma^4\beta^{*2}c\left(\frac{K}{\gamma}\right)^2\left(\frac{2\pi}{\lambda_{ID}}\right)^2\frac{1}{2}$$

Total energy radiated after one passage of the insertion device

$$\delta E = 2\pi^2\frac{e^2}{6\pi\varepsilon_0\lambda_{ID}}\gamma^2\beta^* NK^2$$

Power Emitted Beam Frame

Larmor/Lienard calculation in the beam frame yields

$$\left\langle P^* \right\rangle = \frac{2e^2}{6\pi\varepsilon_0} cK^2 \left(\frac{2\pi}{\lambda^*} \right)^2 \frac{1}{2}$$

Total energy of each photon is $\hbar 2\pi c/\lambda^*$, therefore the total number of photons radiated after one passage of the insertion device

$$N_\gamma = \frac{2\pi}{3} \alpha N K^2$$

Spectral Distribution in Beam Frame

Begin with power distribution in beam frame: dipole radiation pattern (single harmonic only when K<<1; replace γ* by γ, β* by β)

$$\frac{dP^*}{d\Omega^*} = \frac{e^2 c}{32\pi^2 \varepsilon_0} k^{*4} a^2 \sin^2 \Theta^*$$

Number distribution in terms of wave number

$$\frac{dN_\gamma}{d\Omega^*} = \frac{\alpha}{4} N K^2 \frac{k_y^{*2} + k_z^{*2}}{k^{*2}}$$

Solid angle transformation

$$d\Omega^* = \frac{d\Omega}{\gamma^2 \left(1 - \beta \cos\theta\right)^2}$$

Number distribution in beam frame

$$\frac{dN_\gamma}{d\Omega} = \frac{\alpha}{4} NK^2 \frac{\sin^2\theta \sin^2\varphi + \gamma^2 \left(\cos\theta - \beta\right)^2}{\gamma^4 \left(1 - \beta\cos\theta\right)^4}$$

Energy is simply

$$E(\theta) = \hbar \frac{2\pi\beta c}{\lambda_{ID}\left(1 - \beta\cos\theta\right)} \qquad \hat{E}(\theta) = \frac{1}{\left(1 - \beta\cos\theta\right)}$$

Number distribution as a function of normalized lab-frame energy

$$\frac{dN_\gamma}{d\hat{E}} = \frac{\alpha\pi}{4\gamma^2\beta^3} NK^2 \left[\left(\frac{\hat{E}}{\gamma^2} - 1\right)^2 + \beta^2\right]$$

Average Energy

Limits of integration

$$\cos\theta = 1 \qquad \hat{E} = \frac{1}{1-\beta} \qquad \cos\theta = -1 \qquad \hat{E} = \frac{1}{1+\beta}$$

Average energy is also analytically calculable

$$\langle E \rangle = \frac{\int_0^\infty E \frac{dN_\gamma}{d\hat{E}} d\hat{E}}{\int_0^\infty \frac{dN_\gamma}{d\hat{E}} d\hat{E}} = \gamma^2 \hbar 2\pi\beta c / \lambda_{ID} \approx \frac{E_{\max}}{2}$$

Beam *rms* Emittance

Treat the distribution of particles within the beam statistically.
Define single particle distribution function

$$\rho(x, x')$$

And statistical averaging

$$\langle q \rangle \equiv \int q(x, x')\rho(x, x')\, dx dx'$$

Closest *rms* Fit Ellipses

For zero-centered distributions, i.e., distributions that have zero average value for x and x'

$$\varepsilon_{rms} \equiv \sqrt{\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2}$$

$$\beta = \frac{\left\langle x^2 \right\rangle}{\varepsilon_{rms}} = \frac{\sigma_x^2}{\varepsilon_{rms}}$$

$$\alpha = -\frac{\left\langle xx' \right\rangle}{\varepsilon_{rms}}$$

$$\gamma = \frac{\left\langle x'^2 \right\rangle}{\varepsilon_{rms}} = \frac{\sigma_{x'}^2}{\varepsilon_{rms}}$$

Case: Uniformly Filled Ellipse

$$\rho(x,x') = \frac{1}{\pi\varepsilon}\Theta\left(1 - \frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{\varepsilon}\right)$$

Θ here is the Heavyside step function, 1 for positive values of its argument and zero for negative values for its argument

$$\sigma_x^2 = \langle x^2 \rangle = \frac{\varepsilon\beta}{4}$$

$$\langle x x' \rangle = -\frac{\varepsilon\alpha}{4}$$

$$\sigma_{x'}^2 = \langle x'^2 \rangle = \frac{\varepsilon}{4\beta}\left(1+\alpha^2\right)$$

$$\therefore \varepsilon_{rms} = \frac{\varepsilon}{4}$$