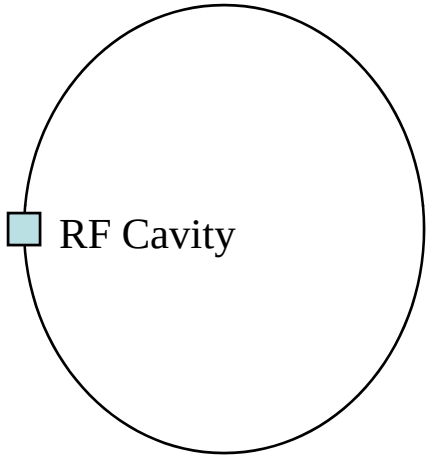
Physics 417/517 Introduction to Particle Accelerator Physics

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

Synchrotrons

Two basic generalizations needed

- Acceleration of non-relativistic particles
- Difference equation describing per turn dynamics becomes a differential equation with solution involving a new frequency, the synchrotron frequency



Acceleration of non-relativistic particles

For microtron, racetrack microtron and other polytrons, electron speed is at the speed of light. For non-relativistic particles the recirculation time also depends on the longitudinal velocity $v_z = \beta_z c$.

$$t_{recirc} = L / \beta_z c$$

$$\Delta t = \frac{\partial L}{\partial p}\frac{\Delta p}{\beta_z c} + \frac{L}{c}\frac{\partial}{\partial p}\left[\frac{1}{\beta_z}\right]\Delta p$$

$$\frac{\Delta t}{t_{recirc}} = \frac{M_{56}}{L}\frac{\Delta p}{p} - \frac{\Delta \beta_z}{\beta_z} = \frac{M_{56}}{L}\frac{\Delta p}{p} - \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

Momentum Compaction $\alpha = \left(\Delta L / L\right) / \left(\Delta p / p\right) = M_{56} / L$

$$\frac{\Delta t}{t_{recirc}} = -\eta_c \frac{\Delta p}{p} \rightarrow \eta_c = \frac{1}{\gamma^2} - \frac{M_{56}}{L} = \frac{1}{\gamma^2} - \alpha$$

$$2p\Delta pc^2 = 2E\Delta E \rightarrow \frac{\Delta p}{p} = \frac{1}{\beta_z^2}\frac{\Delta E}{E} \rightarrow \frac{\Delta t}{t_{recirc}} = -\frac{\eta_c}{\beta_z^2}\frac{\Delta E}{E}$$

Transition Energy: Energy at which the change in the once around time becomes independent of momentum (energy)

$$\eta_c = 0 \rightarrow \frac{1}{\gamma_t^2} = \frac{M_{56}}{L} = \alpha$$

No Phase Focusing at this energy!

Equation for Synchrotron Oscillations

$$\begin{pmatrix} \Delta\phi_{l+1} \\ \Delta E_{l+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -eV_c \sin\phi_s & 1 \end{pmatrix} \begin{pmatrix} 1 & -\dfrac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\dfrac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} \\ -eV_c \sin\phi_s & 1 + \dfrac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} eV_c \sin\phi_s \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

Assume momentum slowly changing (adiabatic acceleration)
Phase advance per turn is

$$\cos\Delta\mu = 1 + \frac{\pi L\eta_c}{\lambda\beta_z^2 E_l} eV_c \sin\phi_s \rightarrow \Delta\mu^2 \approx -\frac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} eV_c \sin\phi_s$$

So change in phase per unit time is

$$\frac{\Delta\mu}{T_0} \approx \frac{1}{T_0}\sqrt{-\frac{2\pi L\eta_c}{\lambda\beta_z pc} eV_c \sin\phi_s}$$

yielding synchrotron oscillations with frequency

$$\omega_s = \omega_{rev}\sqrt{-\frac{h\eta_c}{2\pi}\frac{eV_c}{pc}\sin\phi_s}$$

where the *harmonic number* $h = L / \beta_z \lambda$, gives the integer number of *RF* oscillations in one turn

Phase Stable Acceleration

At energies below transition, $\eta_c > 0$. To achieve acceleration with phase stability need $\phi_s < 0$

$$\therefore \omega_s = \omega_{rev}\sqrt{\frac{h\eta_c}{2\pi}\frac{eV_c}{pc}\sin\left(-\phi_s\right)}$$

At energies above transition, $\eta_c < 0$, which corresponds to the case we're used to from electrons. To achieve acceleration with phase stability need $\phi_s > 0$

$$\therefore \omega_s = \omega_{rev}\sqrt{\frac{h\left(-\eta_c\right)}{2\pi}\frac{eV_c}{pc}\sin\phi_s}$$

Large Amplitude Effects

Can no longer linearize the energy error equation.

$$\Delta\phi_{l+1} = \Delta\phi_l - \frac{2\pi L\eta_c}{\lambda\beta_z^2 E_l}\Delta E_l$$

$$\Delta E_{l+1} = \Delta E_l + eV_c\left(\cos\left(\phi_s + \Delta\phi_l\right) - \cos\phi_s\right)$$

$$\frac{d\Delta\phi}{dt} \approx \frac{\Delta\phi_{l+1} - \Delta\phi_l}{T_0} = -\frac{2\pi\eta_c}{\lambda p}\Delta E$$

$$\frac{d\Delta E}{dt} \approx \frac{\Delta E_{l+1} - \Delta E_l}{T_0} = \frac{eV_c\left(\cos\left(\phi_s + \Delta\phi_l\right) - \cos\phi_s\right)}{T_0}$$

$$\frac{d^2\Delta\phi}{dt^2} = -\frac{2\pi\eta_c}{\lambda p T_0} eV_c\left(\cos\left(\phi_s + \Delta\phi\right) - \cos\phi_s\right)$$

Constant of Motion (Longitudinal "Hamiltonian")

$$\frac{d\Delta\phi}{dt}\frac{d^2\Delta\phi}{dt^2} = -\frac{2\pi\eta_c}{\lambda p T_0} eV_c \frac{d\Delta\phi}{dt}\left(\cos\left(\phi_s + \Delta\phi\right) - \cos\phi_s\right)$$

$$\frac{1}{2}\left(\frac{d\Delta\phi}{dt}\right)^2 = -\frac{2\pi\eta_c}{\lambda p T_0} eV_c \left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right) + C$$

$$H\left(\Delta\phi, T_0\Delta E\right) = \frac{1}{2}\frac{2\pi\eta_c}{\lambda p T_0}\left(T_0\Delta E\right)^2 + eV_c\left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right)$$

Equations of Motion

If neglect the slow (adiabatic) variation of p and T_0 with time, the equations of motion approximately Hamiltonian

$$\frac{d\Delta\phi}{dt} = \frac{\partial H}{\partial\left(T_0 \Delta E\right)} \qquad \frac{d\left(T_0 \Delta E\right)}{dt} = -\frac{\partial H}{\partial \Delta\phi}$$

In particular, the Hamiltonian is a constant of the motion

Kinetic Energy Term

$$T = \frac{1}{2}\frac{2\pi\eta_c T_0}{\lambda p}\left(\Delta E\right)^2$$

Potential Energy Term

$$V = eV_c\left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right)$$

No Acceleration

$$\phi_s = \pm\pi/2 \qquad V = eV_c \cos\Delta\phi$$

$$\frac{d^2\Delta\phi}{dt^2} = \omega_s^2 \sin\Delta\phi$$

Better known as the real pendulum.

With Acceleration

$$\frac{d^2\Delta\phi}{dt^2} = \frac{\omega_s^2}{\sin\phi_s}\left(\cos\left(\phi_s + \Delta\phi\right) - \cos\phi_s\right)$$

$$\frac{1}{2}\left(\frac{d\Delta\phi}{dt}\right)^2 = \frac{\omega_s^2}{\sin\phi_s}\left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right) + C$$

Equation for separatrix yields “fish” diagrams in phase space.
Fixed points at

$$\cos\left(\phi_s + \Delta\phi\right) = \cos\phi_s \qquad \Delta\phi = 0, -2\phi_s$$

Synchrotron Radiation

Accelerated particles emit electromagnetic radiation. Emission from very high energy particles has unique properties for a radiation source. As such radiation was first observed at one of the earliest electron synchrotrons, radiation from high energy particles (mainly electrons) is known generically as synchrotron radiation by the accelerator and HENP communities.

The radiation is highly collimated in the beam direction

From relativity

$$ct' = \gamma ct - \gamma\beta z$$
$$x' = x$$
$$y' = y$$
$$z' = -\gamma\beta ct + \gamma z$$

Lorentz invariance of wave phase implies $k^\mu = (\omega/c, k_x, k_y, k_z)$ is a Lorentz 4-vector

$$\omega' = \gamma\omega - \gamma\beta k_z c$$
$$k'_x = k_x$$
$$k'_y = k_y$$
$$k'_z = -\gamma\beta c\omega + \gamma k_z$$

$$\sin\theta = \frac{\sqrt{k_x^2 + k_y^2}}{\omega / c} \qquad \sin\theta' = \frac{\sqrt{k_x'^2 + k_y'^2}}{\omega' / c} \qquad \cos\theta' = \frac{k'_z}{\omega' / c}$$

$$\omega / c = \gamma\omega' / c + \gamma\beta k'_z = \gamma\left(1 + \beta\cos\theta'\right)\left(\omega' / c\right)$$

$$\theta \approx \sin\theta = \frac{\sin\theta'}{\gamma\left(1+\beta\cos\theta'\right)}$$

Therefore all radiation with $\theta' < \pi / 2$, which is roughly ½ of the emission for dipole emission from a transverse acceleration in the beam frame, is Lorentz transformed into an angle less than $1/\gamma$. Because of the strong Doppler shift of the photon energy, higher for $\theta \rightarrow 0$, most of the energy in the photons is within a cone of angular extent $1/\gamma$ around the beam direction.