# Physics 417/517 Introduction to Particle Accelerator Physics

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# Dispersion Calculation

Begin with the inhomogeneous Hill's equation for the dispersion.

$$\frac{d^2D}{ds^2} + K(s)D = \frac{1}{\rho(s)}$$

Write the general solution to the inhomogeneous equation for the dispersion as before.

$$D(s) = D_p(s) + Ax_1(s) + Bx_2(s)$$

Here $D_p$ can be any particular solution. Suppose that the dispersion and it's derivative are known at the location $s_1$, and we wish to determine their values at $s_2$. $x_1$ and $x_2$, because they are solutions to the homogeneous equations, must be transported by the transfer matrix solution $M_{s2,s1}$ already found.

To build up the general solution, choose that particular solution of the inhomogeneous equation with boundary conditions

$$D_{p,0}\left(s_1\right) = D'_{p,0}\left(s_1\right) = 0$$

Evaluate $A$ and $B$ by the requirement that the dispersion and it's derivative have the proper value at $s_1$ ($x_1$ and $x_2$ need to be linearly independent!)

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} x_1\left(s_1\right) & x_2\left(s_1\right) \\ x'_1\left(s_1\right) & x'_2\left(s_1\right) \end{pmatrix}^{-1} \begin{pmatrix} D\left(s_1\right) \\ D'\left(s_1\right) \end{pmatrix}$$

$$D\left(s_2\right) = D_{p,0}\left(s_2 - s_1\right) + \left(M_{s_2,s_1}\right)_{11} D\left(s_1\right) + \left(M_{s_2,s_1}\right)_{12} D'\left(s_1\right)$$

$$D'\left(s_2\right) = D'_{p,0}\left(s_2 - s_1\right) + \left(M_{s_2,s_1}\right)_{21} D\left(s_1\right) + \left(M_{s_2,s_1}\right)_{22} D'\left(s_1\right)$$

# 3 by 3 Matrices for Dispersion Tracking

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \left(M_{s_2,s_1}\right)_{11} & \left(M_{s_2,s_1}\right)_{12} & D_{p,0}(s_2 - s_1) \\ \left(M_{s_2,s_1}\right)_{21} & \left(M_{s_2,s_1}\right)_{22} & D'_{p,0}(s_2 - s_1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}$$

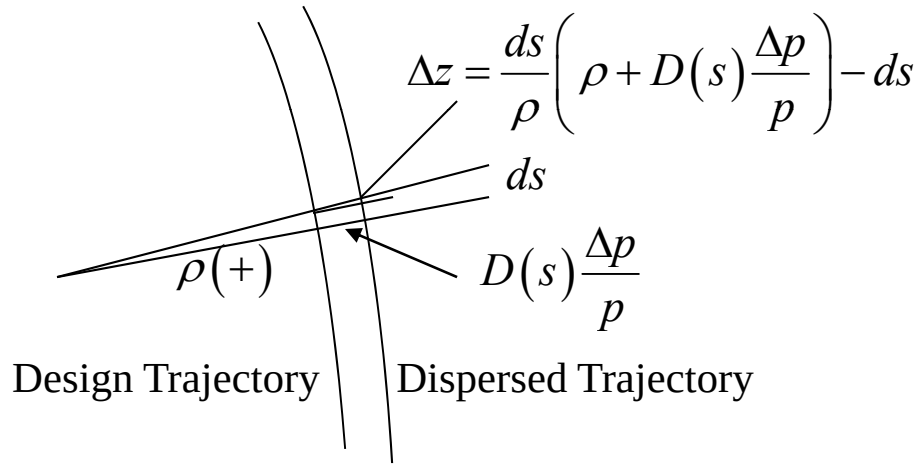
Particular solutions to inhomogeneous equation for constant $K$ and constant $\rho$ and vanishing dispersion and derivative at $s = 0$

| | $K < 0$ | $K = 0$ | $K > 0$ |
|---|---|---|---|
| $D_{p,0}(s)$ | $\frac{1}{\lvert K\rvert\rho}\left(\cosh\left(\sqrt{\lvert K\rvert}s\right)-1\right)$ | $\frac{s^2}{2\rho}$ | $\frac{1}{K\rho}\left(1-\cos\left(\sqrt{K}s\right)\right)$ |
| $D'_{p,0}(s)$ | $\frac{1}{\sqrt{\lvert K\rvert}\rho}\sinh\left(\sqrt{\lvert K\rvert}s\right)$ | $\frac{s}{\rho}$ | $\frac{1}{\sqrt{K}\rho}\sin\left(\sqrt{K}s\right)$ |

# $M_{56}$

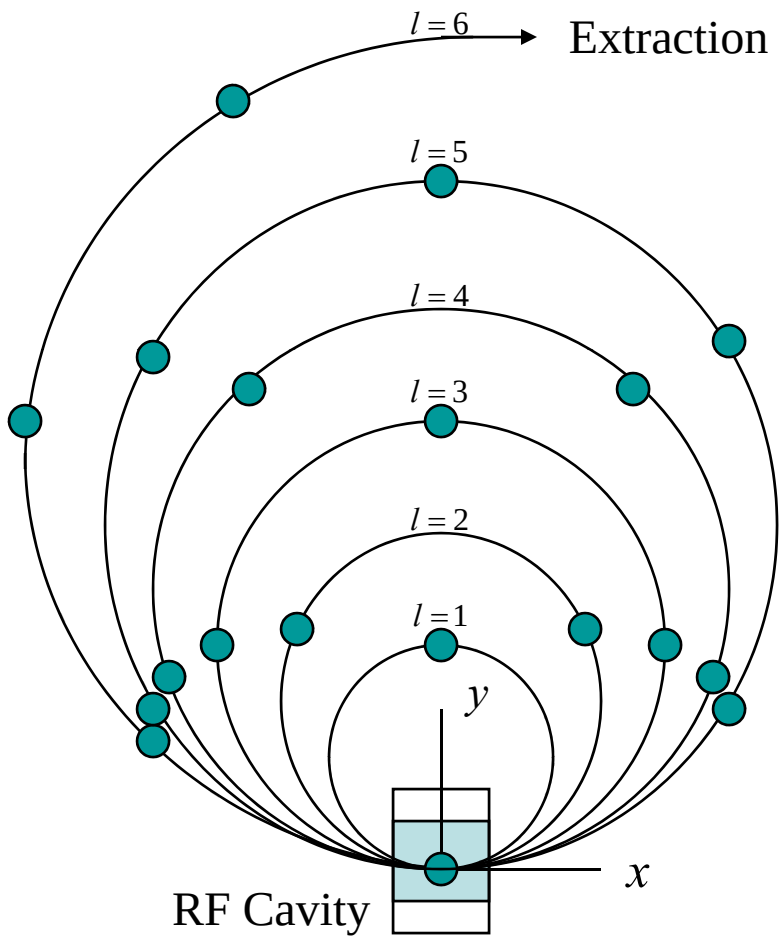
In addition to the transverse effects of the dispersion, there are important effects of the dispersion along the direction of motion. The primary effect is to change the time-of-arrival of the off-momentum particle compared to the on-momentum particle which traverses the design trajectory.

$$d(\Delta z) = D(s)\frac{\Delta p}{p}\frac{ds}{\rho(s)}$$




$$M_{56} = \int_{s_1}^{s_2} \left\{ \frac{D_x(s)}{\rho_x(s)} + \frac{D_y(s)}{\rho_y(s)} \right\} ds$$

# Classical Microtron: Veksler (1945)



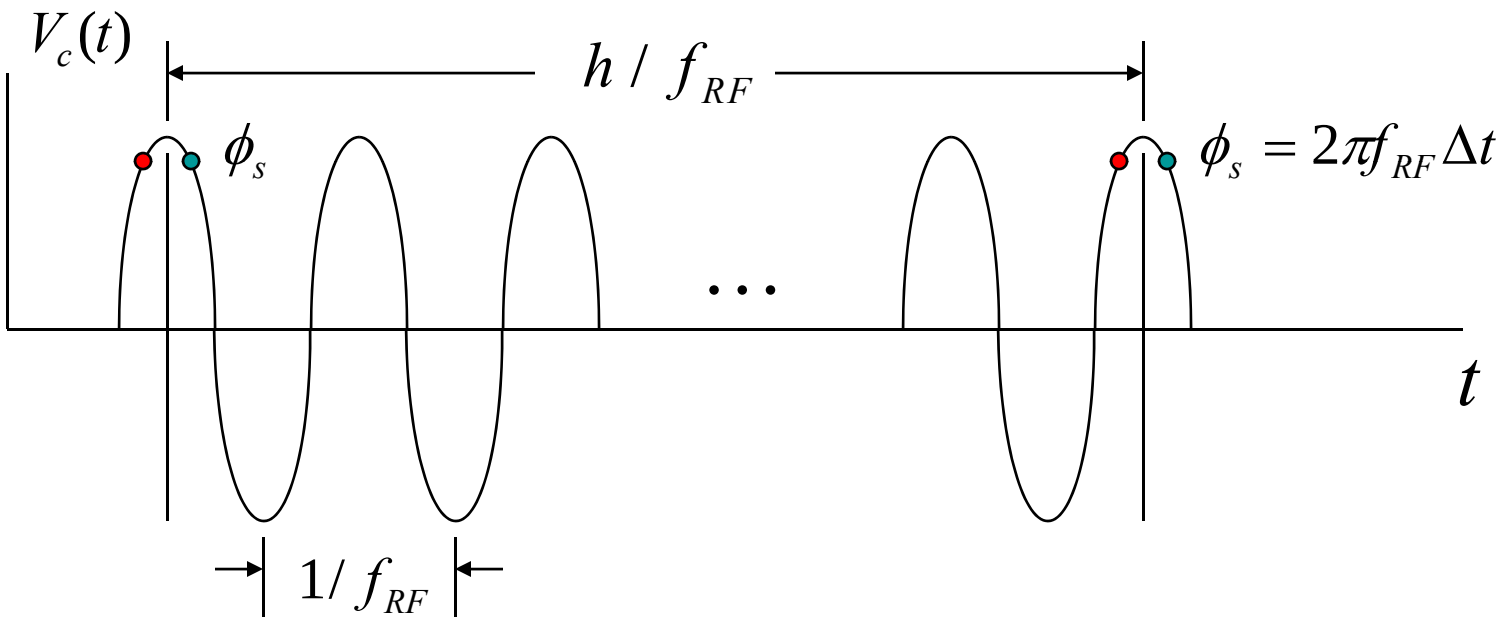

$\otimes$ Magnetic Field

$\mu = 2$

$\nu = 1$

# Synchrotron Phase Stability

Edwin McMillan discovered phase stability independently of Veksler and used the idea to design first large electron synchrotron.




$$h = L f_{RF} / \beta c$$

Harmonic number: # of RF oscillations in a revolution

# Transition Energy

Beam energy where speed increment effect balances path length change effect on accelerator revolution frequency. Revolution frequency independent of beam energy to linear order. We will calculate in a few weeks

- Below Transistion Energy: Particles arriving EARLY get less acceleration and speed increment, and arrive later, with repect to the center of the bunch, on the next pass. Applies to heavy particle synchrotrons during first part of acceleration when the beam is non-relativistic and accelerations still produce velocity changes.

- Above Transistion Energy: Particles arriving EARLY get more energy, have a longer path, and arrive later on the next pass. Applies for electron synchrotrons and heavy particle synchrotrons when approach relativistic velocities. As seen before, Microtrons operate here.

# Phase Stability Condition

"Synchronous" electron has

$$\text{Phase} = \phi_s \qquad\qquad E_l = E_o + leV_c \cos\phi_s$$

Difference equation for differences after passing through cavity pass $l + 1$:

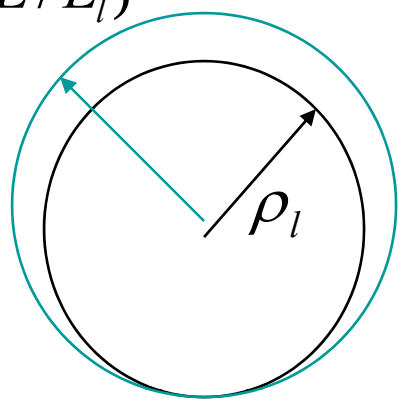
$$\begin{pmatrix} \Delta\phi_{l+1} \\ \Delta E_{l+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -eV_c \sin\phi_s & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{2\pi M_{56}}{\lambda E_l} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

Because for an electron passing the cavity

$$\Delta E_{after} = \Delta E_{before} + eV_c\left(\cos(\phi_s + \Delta\phi) - \cos\phi_s\right)$$

# Phase Stability Condition

$\rho_l(1+\Delta E/E_l)$




$$K_i = 1/\rho_i^2$$

$$D_{x,p,0} = \rho_i\left(1-\cos\left(s/\rho_i\right)\right) \qquad 0 \le s \le 2\pi\rho_i$$

$$\therefore\ M_{56} = \int \frac{D}{\rho} ds = \int_0^{2\pi\rho_l} \left(1-\cos s/\rho_l\right) ds$$

$$= 2\pi\rho_l$$

$$\begin{pmatrix} \Delta\phi_{l+1} \\ \Delta E_{l+1} \end{pmatrix} \approx \begin{pmatrix} 1 & \dfrac{4\pi^2\rho_l}{\lambda E_l} \\ -eV_c\sin\phi_s & 1-\dfrac{4\pi^2\rho_l eV_c}{\lambda E_l}\sin\phi_s \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

# Phase Stability Condition

Have Phase Stability if

$$-1 < \left( \frac{\mathrm{Tr}\, M}{2} \right) < 1 \rightarrow -1 < 1 - \frac{2\pi^2 \rho_l e V_c}{\lambda E_l} \sin\phi_s < 1$$

$$\frac{2\pi^2 \rho_l e V_c}{\lambda E_l} \sin\phi_s = \frac{\pi f_{RF} e V_c}{f_c m c^2} \cos\phi_s \tan\phi_s = \frac{\pi f_{RF} \Delta\gamma}{f_c} \tan\phi_s$$

i.e.,

$$\boxed{0 < \nu\pi \tan\phi_s < 2}$$

# Phase Stability Condition

Have Phase Stability if

$$\left(\frac{\mathrm{Tr}\, M}{2}\right)^2 < 1$$

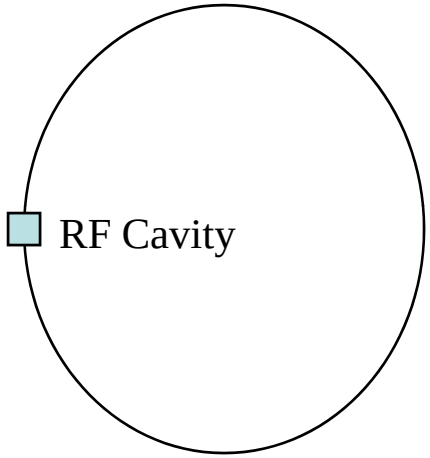
i.e.,

$$\boxed{0 < \nu\pi \tan\phi_s < 2}$$

# Synchrotrons

Two basic generalizations needed

- Acceleration of non-relativistic particles
- Difference equation describing per turn dynamics becomes a differential equation with solution involving a new frequency, the synchrotron frequency



# Acceleration of non-relativistic particles

For microtron, racetrack microtron and other polytrons, electron speed is at the speed of light. For non-relativistic particles the recirculation time also depends on the longitudinal velocity $v_z = \beta_z c$.

$$t_{recirc} = L / \beta_z c$$

$$\Delta t = \frac{\partial L}{\partial p}\frac{\Delta p}{\beta_z c} + \frac{L}{c}\frac{\partial}{\partial p}\left[\frac{1}{\beta_z}\right]\Delta p$$

$$\frac{\Delta t}{t_{recirc}} = \frac{M_{56}}{L}\frac{\Delta p}{p} - \frac{\Delta \beta_z}{\beta_z} = \frac{M_{56}}{L}\frac{\Delta p}{p} - \frac{1}{\gamma^2}\frac{\Delta p}{p}$$