Physics 417/517 Introduction to Particle Accelerator Physics

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Courant-Snyder Invariant

Consider now a single particular solution of the equations of motion generated by Hill's equation. We've seen that once a particle is on an invariant ellipse for a period, it must stay on that ellipse throughout its motion. Because the phase space area of the single period invariant ellipse is preserved by the motion, the quantity that gives the phase space area of the invariant ellipse in terms of the single particle orbit must also be an invariant. This phase space area/π,

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \left(x^2 + \left(\beta x' + \alpha x\right)^2\right) / \beta$$

is called the Courant-Snyder invariant. It may be verified to be a constant by showing its derivative with respect to s is zero by Hill's equation, or by explicit substitution of the transfer matrix solution which begins at some initial value $s = 0$.

Pseudoharmonic Solution

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}\left(\cos\Delta\mu_{s,0} + \alpha_0 \sin\Delta\mu_{s,0}\right) & \sqrt{\beta(s)\beta_0}\sin\Delta\mu_{s,0} \\ -\frac{1}{\sqrt{\beta(s)\beta_0}}\begin{bmatrix}(1+\alpha(s)\alpha_0)\sin\Delta\mu_{s,0} \\ +(\alpha(s)-\alpha_0)\cos\Delta\mu_{s,0}\end{bmatrix} & \sqrt{\frac{\beta_0}{\beta(s)}}\left(\cos\Delta\mu_{s,0} - \alpha(s)\sin\Delta\mu_{s,0}\right) \end{pmatrix} \begin{pmatrix} x_0 \\ \left.\frac{dx}{ds}\right|_0 \end{pmatrix}$$

gives

$$\left(x^2(s) + \left(\beta(s)x'(s) + \alpha(s)x(s)\right)^2\right)/\beta(s) = \left(x_0^2 + \left(\beta_0 x'_0 + \alpha_0 x_0\right)^2\right)/\beta_0 \equiv \varepsilon$$

Using the $x(s)$ equation above and the definition of ε, the solution may be written in the standard "pseudoharmonic" form

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos\left(\Delta\mu_{s,0} - \delta\right) \quad \text{where } \delta = \tan^{-1}\left(\frac{\beta_0 x'_0 + \alpha_0 x_0}{x_0}\right)$$

The the origin of the terminology "phase advance" is now obvious.

Case II: $K(s)$ not periodic

In a linac or a recirculating linac there is no closed orbit or natural machine periodicity. Designing the transverse optics consists of arranging a focusing lattice that assures the beam particles coming into the front end of the accelerator are accelerated (and sometimes decelerated!) with as small beam loss as is possible. Therefore, it is imperative to know the initial beam phase space injected into the accelerator, in addition to the transfer matrices of all the elements making up the focusing lattice of the machine. An initial ellipse, or a set of initial conditions that somehow bound the phase space of the injected beam, are tracked through the acceleration system element by element to determine the transmission of the beam through the accelerator. The designs are usually made up of well-understood "modules" that yield known and understood transverse beam optical properties.

Definition of β function

Now the pseudoharmonic solution applies even when $K(s)$ is not periodic. Suppose there is an ellipse, the design injected ellipse, which tightly includes the phase space of the beam at injection to the accelerator. Let the ellipse parameters for this ellipse be α_0, β_0, and γ_0. A function $\beta(s)$ is simply defined by the ellipse transformation rule

$$\begin{aligned}\beta(s) &= \left(M_{12}(s)\right)^2\gamma_0 - 2M_{12}(s)M_{11}(s)\alpha_0 + \left(M_{11}(s)\right)^2\beta_0 \\ &= \left[\left(M_{12}(s)\right)^2 + \left(\beta_0 M_{11}(s) - \alpha_0 M_{12}(s)\right)^2\right]/\beta_0\end{aligned}$$

where

$$M_{s,0} \equiv \begin{pmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{pmatrix}$$

One might think to evaluate the phase advance by integrating the beta-function. Generally, it is far easier to evaluate the phase advance using the general formula,

$$\tan \Delta\mu_{s',s} = \frac{\left(M_{s',s}\right)_{12}}{\beta(s)\left(M_{s',s}\right)_{11} - \alpha(s)\left(M_{s',s}\right)_{12}}$$

where $\beta(s)$ and $\alpha(s)$ are the ellipse functions at the entrance of the region described by transport matrix $M_{s',s}$. Applied to the situation at hand yields

$$\tan \Delta\mu_{s,0} = \frac{M_{12}(s)}{\beta_0 M_{11}(s) - \alpha_0 M_{12}(s)}$$

Beam Matching

Fundamentally, in circular accelerators beam matching is applied in order to guarantee that the beam envelope of the real accelerator beam does not depend on time. This requirement is one part of the definition of having a stable beam. With periodic boundary conditions, this means making beam density contours in phase space align with the invariant ellipses (in particular at the injection location!) given by the ellipse functions. Once the particles are on the invariant ellipses they stay there (in the linear approximation!), and the density is preserved because the single particle motion is around the invariant ellipses. In linacs and recirculating linacs, usually different purposes are to be achieved. If there are regions with periodic focusing lattices within the linacs, matching as above ensures that the beam

envelope does not grow going down the lattice. Sometimes it is advantageous to have specific values of the ellipse functions at specific longitudinal locations. Other times, re/matching is done to preserve the beam envelopes of a good beam solution as changes in the lattice are made to achieve other purposes, e.g. changing the dispersion function or changing the chromaticity of regions where there are bends (see the next chapter for definitions). At a minimum, there is usually a matching done in the first parts of the injector, to take the phase space that is generated by the particle source, and change this phase space in a way towards agreement with the nominal transverse focusing design of the rest of the accelerator. The ellipse transformation formulas, solved by computer, are essential for performing this process.

Dispersion Calculation

Begin with the inhomogeneous Hill's equation for the dispersion.

$$\frac{d^2 D}{ds^2} + K(s) D = \frac{1}{\rho(s)}$$

Write the general solution to the inhomogeneous equation for the dispersion as before.

$$D(s) = D_p(s) + A x_1(s) + B x_2(s)$$

Here D_p can be any particular solution. Suppose that the dispersion and it's derivative are known at the location s_1, and we wish to determine their values at s_2. x_1 and x_2, because they are solutions to the homogeneous equations, must be transported by the transfer matrix solution $M_{s2,s1}$ already found.

To build up the general solution, choose that particular solution of the inhomogeneous equation with boundary conditions

$$D_{p,0}\left(s_1\right) = D'_{p,0}\left(s_1\right) = 0$$

Evaluate A and B by the requirement that the dispersion and it's derivative have the proper value at s_1 (x_1 and x_2 need to be linearly independent!)

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} x_1\left(s_1\right) & x_2\left(s_1\right) \\ x'_1\left(s_1\right) & x'_2\left(s_1\right) \end{pmatrix}^{-1} \begin{pmatrix} D\left(s_1\right) \\ D'\left(s_1\right) \end{pmatrix}$$

$$D\left(s_2\right) = D_{p,0}\left(s_2 - s_1\right) + \left(M_{s_2,s_1}\right)_{11} D\left(s_1\right) + \left(M_{s_2,s_1}\right)_{12} D'\left(s_1\right)$$

$$D'\left(s_2\right) = D'_{p,0}\left(s_2 - s_1\right) + \left(M_{s_2,s_1}\right)_{21} D\left(s_1\right) + \left(M_{s_2,s_1}\right)_{22} D'\left(s_1\right)$$

3 by 3 Matrices for Dispersion Tracking

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \left(M_{s_2,s_1}\right)_{11} & \left(M_{s_2,s_1}\right)_{12} & D_{p,0}(s_2 - s_1) \\ \left(M_{s_2,s_1}\right)_{21} & \left(M_{s_2,s_1}\right)_{22} & D'_{p,0}(s_2 - s_1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}$$

Particular solutions to inhomogeneous equation for constant K and constant ρ and vanishing dispersion and derivative at $s = 0$

	$K < 0$	$K = 0$	$K > 0$
$D_{p,0}(s)$	$\frac{1}{\lvert K\rvert\rho}\left(\cosh\left(\sqrt{\lvert K\rvert}s\right)-1\right)$	$\frac{s^2}{2\rho}$	$\frac{1}{K\rho}\left(1-\cos\left(\sqrt{K}s\right)\right)$
$D'_{p,0}(s)$	$\frac{1}{\sqrt{\lvert K\rvert}\rho}\sinh\left(\sqrt{\lvert K\rvert}s\right)$	$\frac{s}{\rho}$	$\frac{1}{\sqrt{K}\rho}\sin\left(\sqrt{K}s\right)$