Physics 417/517 Introduction to Particle Accelerator Physics

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Phase Advance of a Unimodular Matrix

Any two-by-two unimodular (Det (M) = 1) matrix with $|\mathrm{Tr}\, M| < 2$ can be written in the form

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

The *phase advance* of the matrix, μ, gives the eigenvalues of the matrix $\lambda = e^{\pm i\mu}$, and $\cos \mu = (\mathrm{Tr}\, M)/2$. Furthermore $\beta\gamma - \alpha^2 = 1$

Pf: The equation for the eigenvalues of M is

$$\lambda^2 - (M_{11} + M_{22})\lambda + 1 = 0$$

Because M is real, both λ and λ^* are solutions of the quadratic. Because

$$\lambda = \frac{\mathrm{Tr}(M)}{2} \pm i\sqrt{1-(\mathrm{Tr}(M)/2)^2}$$

For $|\mathrm{Tr}\, M| < 2$, $\lambda\,\lambda^* = 1$ and so $\lambda_{1,2} = e^{\pm i\mu}$. Consequently $\cos\mu = (\mathrm{Tr}\, M)/2$. Now the following matrix is trace-free.

$$M - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\mu) = \begin{pmatrix} \frac{M_{11}-M_{22}}{2} & M_{12} \\ M_{21} & \frac{M_{22}-M_{11}}{2} \end{pmatrix}$$

Simply choose

$$\alpha = \frac{M_{11} - M_{22}}{2\sin\mu}, \qquad \beta = \frac{M_{12}}{\sin\mu}, \qquad \gamma = -\frac{M_{21}}{\sin\mu}$$

and the sign of μ to properly match the individual matrix elements with $\beta > 0$. It is easily verified that $\beta\gamma - \alpha^2 = 1$. Now

$$M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(2\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(2\mu)$$

and more generally

$$M^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(n\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(n\mu)$$

Therefore, because sin and cos are both bounded functions, the matrix elements of any power of M remain bounded as long as $|\mathrm{Tr}\,(M)| < 2$.

NB, in some beam dynamics literature it is (incorrectly!) stated that the less stringent $|\mathrm{Tr}\,(M)| \leq 2$ ensures boundedness and/or stability. That equality cannot be allowed can be immediately demonstrated by counterexample. The upper triangular or lower triangular subgroups of the two-by-two unimodular matrices, i.e., matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

clearly have unbounded powers if $|x|$ is not equal to 0.

Significance of matrix parameters

Another way to interpret the parameters α, β, and γ, which represent the unimodular matrix M (these parameters are sometimes called the Twiss parameters or Twiss representation for the matrix) is as the "coordinates" of that specific set of ellipses that are mapped onto each other, or are invariant, under the linear action of the matrix. This result is demonstrated in

Thm: For the unimodular linear transformation

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

with $|\mathrm{Tr}\,(M)| < 2$, the ellipses

$$\gamma x^2 + 2\alpha xy + \beta y^2 = c$$

are invariant under the linear action of M, where c is any constant. Furthermore, these are the only invariant ellipses. Note that the theorem does not apply to $\pm I$, because $|\mathrm{Tr}\,(\pm I)| = 2$.

Pf: The inverse to M is clearly

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) - \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

By the ellipse transformation formulas, for example

$$\beta' = \beta^2 \left(\sin^2 \mu\right)\gamma + 2\left(-\beta \sin \mu\right)\left(\cos \mu + \alpha \sin \mu\right)\alpha + \left(\cos \mu + \alpha \sin \mu\right)^2 \beta$$

$$= \beta \sin^2 \mu \left(1 + \alpha^2\right) - 2\beta\alpha^2 \sin^2 \mu + \beta \cos^2 \mu + \beta\alpha^2 \sin^2 \mu$$

$$= \left(\sin^2 \mu + \cos^2 \mu\right)\beta = \beta$$

Similar calculations demonstrate that $\alpha' = \alpha$ and $\gamma' = \gamma$. As det $(M) = 1$, $c' = c$, and therefore the ellipse is invariant. Conversely, suppose that an ellipse is invariant. By the ellipse transformation formula, the specific ellipse

$$\gamma_i x^2 + 2\alpha_i xy + \beta_i y^2 = \varepsilon$$

is invariant under the transformation by M only if

$$\begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} (\cos\mu - \alpha\sin\mu)^2 & 2(\cos\mu - \alpha\sin\mu)(\gamma\sin\mu) & (\gamma\sin\mu)^2 \\ -(\cos\mu - \alpha\sin\mu)(\beta\sin\mu) & 1 - 2\beta\gamma\sin^2\mu & (\cos\mu + \alpha\sin\mu)(\gamma\sin\mu) \\ (\beta\sin\mu)^2 & -2(\cos\mu + \alpha\sin\mu)(\beta\sin\mu) & (\cos\mu + \alpha\sin\mu)^2 \end{pmatrix} \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix}$$

$$\equiv T_M \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} \equiv T_M \vec{v},$$

i.e., if the vector $\vec{v}$ is *ANY* eigenvector of T_M with eigenvalue 1. All possible solutions may be obtained by investigating the eigenvalues and eigenvectors of T_M. Now

$$T_M \vec{v}_\lambda = \lambda \vec{v}_\lambda \quad \text{has a solution when Det}\left(T_M - \lambda I\right) = 0$$

i.e.,

$$\left(\lambda^2 + \left[2 - 4\cos^2\mu\right]\lambda + 1\right)\left(1 - \lambda\right) = 0$$

Therefore, M generates a transformation matrix T_M with at least one eigenvalue equal to 1. For there to be more than one solution with $\lambda = 1$,

$$1 + \left[2 - 4\cos^2\mu\right] + 1 = 0, \quad \cos^2\mu = 1, \quad \text{or } M = \pm I$$

and we note that *all* ellipses are invariant when $M = \pm I$. But, these two cases are excluded by hypothesis. Therefore, M generates a transformation matrix T_M which always possesses a single nondegenerate eigenvalue 1; the set of eigenvectors corresponding to the eigenvalue 1, all proportional to each other, are the only vectors whose components (γ_i, α_i, β_i) yield equations for the invariant ellipses. For concreteness, compute that eigenvector with eigenvalue 1 normalized so $\beta_i\gamma_i - \alpha_i^2 = 1$

$$\vec{v}_{1,i} = \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} = \beta \begin{pmatrix} -M_{21}/M_{12} \\ (M_{11} - M_{22})/2M_{12} \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma \\ \alpha \\ \beta \end{pmatrix}$$

All other eigenvectors with eigenvalue 1 have $\vec{v}_1 = \varepsilon \vec{v}_{1,i} / c$, for some value c.

Because Det (M) =1, the eigenvector $\vec{v}_{1,i}$ clearly yields the invariant ellipse

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon.$$

Likewise, the proportional eigenvector $\vec{v}_1$ generates the similar ellipse

$$\frac{\varepsilon}{c}\left(\gamma x^2 + 2\alpha xy + \beta y^2\right) = \varepsilon$$

Because we have enumerated all possible eigenvectors with eigenvalue 1, all ellipses invariant under the action of M, are of the form

$$\gamma x^2 + 2\alpha xy + \beta y^2 = c$$

To summarize, this theorem gives a way to tie the mathematical representation of a unimodular matrix in terms of its α, β, and γ, and its phase advance, to the equations of the ellipses invariant under the matrix transformation. The equations of the invariant ellipses when properly normalized have precisely the same α, β, and γ as in the Twiss representation of the matrix, but varying c.

Finally note that throughout this calculation c acts merely as a scale parameter for the ellipse. All ellipses similar to the starting ellipse, i.e., ellipses whose equations have the same α, β, and γ, but with different c, are also invariant under the action of M. Later, it will be shown that more generally

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \left(x^2 + \left(\beta x' + \alpha x\right)^2\right) / \beta$$

is an invariant of the equations of transverse motion.

Applications to transverse beam optics

When the motion of particles in transverse phase space is considered, linear optics provides a good first approximation of the transverse particle motion. Beams of particles are represented by ellipses in phase space (i.e. in the (x, x') space). To the extent that the transverse forces are linear in the deviation of the particles from some pre-defined central orbit, the motion may analyzed by applying ellipse transformation techniques.

Transverse Optics Conventions: positions are measured in terms of length and angles are measured by radian measure. The area in phase space divided by π, ε, measured in m-rad, is called the emittance. In such applications, α has no units, β has units m/radian. Codes that calculate β, by widely accepted convention, drop the per radian when reporting results, it is implicit that the units for x' are radians.

Linear Transport Matrix

Within a linear optics description of transverse particle motion, the particle transverse coordinates at a location s along the beam line are described by a vector

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

If the differential equation giving the evolution of x is linear, one may define a linear transport matrix $M_{s',s}$ relating the coordinates at s' to those at s by

$$\begin{pmatrix} x(s') \\ \frac{dx}{ds}(s') \end{pmatrix} = M_{s',s} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

From the definitions, the concatenation rule $M_{s'',s} = M_{s'',s'} M_{s',s}$ must apply for all s' such that $s < s' < s''$ where the multiplication is the usual matrix multiplication.

Pf: The equations of motion, linear in x and dx/ds, generate a motion with

$$M_{s'',s}\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} x(s'') \\ \frac{dx}{ds}(s'') \end{pmatrix} = M_{s'',s'}\begin{pmatrix} x(s') \\ \frac{dx}{ds}(s') \end{pmatrix} = M_{s'',s'} M_{s',s}\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

for all initial conditions ($x(s)$, $dx/ds(s)$), thus $M_{s'',s} = M_{s'',s'} M_{s',s}$.

Clearly $M_{s,s} = I$. As is shown next, the matrix $M_{s',s}$ is in general a member of the unimodular subgroup of the general linear group.

Ellipse Transformations Generated by Hill's Equation

The equation governing the linear transverse dynamics in a particle accelerator, without acceleration, is *Hill's equation**

$$\frac{d^2x}{ds^2} + K(s)x = 0 \qquad \text{Eqn. (2)}$$

The transformation matrix taking a solution through an infinitesimal distance *ds* is

$$\begin{pmatrix} x(s+ds) \\ \frac{dx}{ds}(s+ds) \end{pmatrix} = \begin{pmatrix} 1 & \frac{ds}{\text{rad}} \\ -K(s)ds\ \text{rad} & 1 \end{pmatrix} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} \equiv M_{s+ds,s} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

* Strictly speaking, Hill studied Eqn. (2) with periodic *K*. It was first applied to circular accelerators which had a periodicity given by the circumference of the machine. It is a now standard in the field of beam optics, to still refer to Eqn. 2 as Hill's equation, even in cases, as in linear accelerators, where there is no periodicity.

Suppose we are given the phase space ellipse

$$\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 = \varepsilon$$

at location s, and we wish to calculate the ellipse parameters, after the motion generated by Hill's equation, at the location $s + ds$

$$\gamma(s+ds)x^2 + 2\alpha(s+ds)xx' + \beta(s+ds)x'^2 = \varepsilon'$$

Because, to order linear in ds, Det $M_{s+ds,s} = 1$, at all locations s, $\varepsilon' = \varepsilon$, and thus the phase space area of the ellipse after an infinitesimal displacement must equal the phase space area before the displacement. Because the transformation through a finite interval in s can be written as a series of infinitesimal displacement transformations, all of which preserve the phase space area of the transformed ellipse, we come to two important conclusions:

1. The phase space area is preserved after a finite integration of Hill's equation to obtain $M_{s',s}$, the transport matrix which can be used to take an ellipse at s to an ellipse at s'. This conclusion holds generally for all s' and s.

2. Therefore Det $M_{s',s}$ = 1 for all s' and s, independent of the details of the functional form $K(s)$. (If desired, these two conclusions may be verified more analytically by showing that

$$\frac{d}{ds}\left(\beta\gamma - \alpha^2\right) = 0 \quad \rightarrow \quad \beta(s)\gamma(s) - \alpha^2(s) = 1, \quad \forall s$$

may be derived directly from Hill's equation.)

Evolution equations for the α, β functions

The ellipse transformation formulas give, to order linear in ds

$$\beta(s+ds) = -2\alpha\frac{ds}{\text{rad}} + \beta(s)$$

$$\alpha(s+ds) = -\gamma(s)\frac{ds}{\text{rad}} + \alpha(s) + \beta(s)Kds\ \text{rad}$$

So

$$\frac{d\beta}{ds}(s) = -\frac{2\alpha(s)}{\text{rad}}$$

$$\frac{d\alpha}{ds}(s) = \beta(s)K\ \text{rad} - \frac{\gamma(s)}{\text{rad}}$$