Physics 417/517 Introduction to Particle Accelerator Physics

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

The area is easily computed to be

$$\frac{\text{Area}}{\pi} \equiv \varepsilon = \frac{D}{\sqrt{AC - B^2}} \qquad \text{Eqn. (1)}$$

So the equation is equivalently

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon$$

$$\gamma = \frac{A}{\sqrt{AC - B^2}}, \quad \alpha = \frac{B}{\sqrt{AC - B^2}}, \quad \text{and} \quad \beta = \frac{C}{\sqrt{AC - B^2}}$$

When normalized in this manner, the equation coefficients clearly satisfy

$$\beta\gamma - \alpha^2 = 1$$

Example: the defining equation for the upright ellipse may be rewritten in following suggestive way

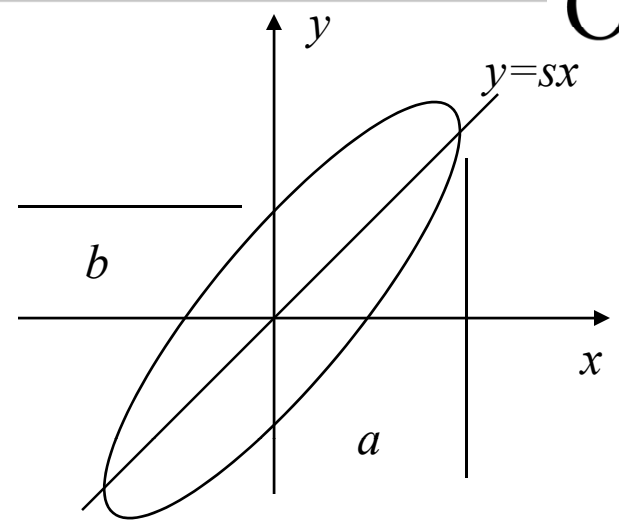
$$\frac{b}{a}x^2 + \frac{a}{b}y^2 = ab = \varepsilon$$

$\beta = a/b$ and $\gamma = b/a$, note $x_{\max} = a = \sqrt{\beta\varepsilon}$, $y_{\max} = b = \sqrt{\gamma\varepsilon}$

General Tilted Ellipse

Needs 3 parameters for a complete description. One way

$$\frac{b}{a}x^2 + \frac{a}{b}(y - sx)^2 = ab = \varepsilon$$



where s is a slope parameter, a is the maximum extent in the x-direction, and the y-intercept occurs at $\pm b$, and again ε is the area of the ellipse divided by π

$$\frac{b}{a}\left(1 + s^2\frac{a^2}{b^2}\right)x^2 - 2s\frac{a}{b}xy + \frac{a}{b}y^2 = ab = \varepsilon$$

Identify

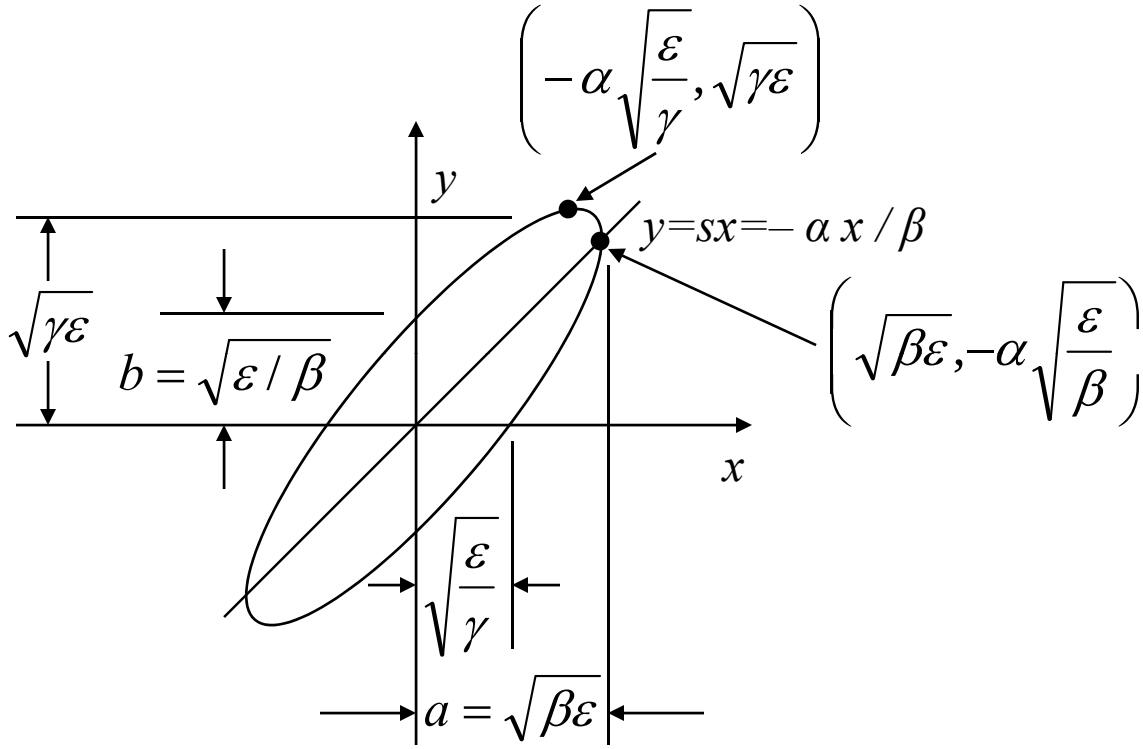
$$\gamma = \frac{b}{a}\left(1 + s^2 \frac{a^2}{b^2}\right), \qquad \alpha = -\frac{a}{b}s, \qquad \beta = \frac{a}{b}$$

Note that $\beta\gamma - \alpha^2 = 1$ automatically, and that the equation for ellipse becomes

$$x^2 + \left(\beta y + \alpha x\right)^2 = \beta\varepsilon$$

by eliminating the (redundant!) parameter γ

Ellipse Dimensions in the β-function Description



As for the upright ellipse $\quad x_{\max} = \sqrt{\beta\varepsilon}, \qquad y_{\max} = \sqrt{\gamma\varepsilon}$

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Area Theorem for Linear Optics

Under a general linear transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

an ellipse is transformed into another ellipse. Furthermore, if det (M) = 1, the area of the ellipse after the transformation is the same as that before the transformation.

Pf: Let the initial ellipse, normalized as above, be

$$\gamma_0 x^2 + 2\alpha_0 xy + \beta_0 y^2 = \varepsilon_0$$

Because

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(M^{-1}\right)_{11} & \left(M^{-1}\right)_{12} \\ \left(M^{-1}\right)_{21} & \left(M^{-1}\right)_{22} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

The transformed ellipse is

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon_0$$

$$\gamma = \left(M^{-1}\right)_{11}^2 \gamma_0 + 2\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{21} \alpha_0 + \left(M^{-1}\right)_{21}^2 \beta_0$$

$$\alpha = \left(M^{-1}\right)_{11}\left(M^{-1}\right)_{12} \gamma_0 + \left(\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{22} + \left(M^{-1}\right)_{12}\left(M^{-1}\right)_{21}\right)\alpha_0 + \left(M^{-1}\right)_{21}\left(M^{-1}\right)_{22} \beta_0$$

$$\beta = \left(M^{-1}\right)_{12}^2 \gamma_0 + 2\left(M^{-1}\right)_{12}\left(M^{-1}\right)_{22} \alpha_0 + \left(M^{-1}\right)_{22}^2 \beta_0$$

Because (verify!)

$$\beta\gamma - \alpha^2 = \left(\beta_0\gamma_0 - \alpha_0^2\right)$$
$$\times\left(\left(M^{-1}\right)_{21}^2\left(M^{-1}\right)_{12}^2 + \left(M^{-1}\right)_{11}^2\left(M^{-1}\right)_{22}^2 - 2\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{22}\left(M^{-1}\right)_{12}\left(M^{-1}\right)_{21}\right)$$
$$= \left(\beta_0\gamma_0 - \alpha_0^2\right)\left(\det M^{-1}\right)^2$$

the area of the transformed ellipse (divided by π) is, by Eqn. (1)

$$\frac{\text{Area}}{\pi} = \varepsilon = \frac{\varepsilon_0}{\sqrt{\beta_0\gamma_0 - \alpha_0^2}\left|\det M^{-1}\right|} = \varepsilon_0 \,|\det M|$$

Tilted ellipse from the upright ellipse

In the tilted ellipse the y-coordinate is raised by the slope with respect to the un-tilted ellipse

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\gamma_0 = \frac{b}{a}, \qquad \alpha_0 = 0, \qquad \beta_0 = \frac{a}{b}, \qquad \left(M^{-1}\right)_{21} = -s$$

$$\therefore \quad \gamma = \frac{b}{a} + \frac{a}{b} s^2, \qquad \alpha = -\frac{a}{b} s, \qquad \beta = \frac{a}{b}$$

Because det (M)=1, the tilted ellipse has the same area as the upright ellipse, i.e., $\varepsilon = \varepsilon_0$.

Phase Advance of a Unimodular Matrix

Any two-by-two unimodular (Det (M) = 1) matrix with $|\mathrm{Tr}\, M| < 2$ can be written in the form

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

The *phase advance* of the matrix, μ, gives the eigenvalues of the matrix $\lambda = e^{\pm i\mu}$, and $\cos \mu = (\mathrm{Tr}\, M)/2$. Furthermore $\beta\gamma - \alpha^2 = 1$

Pf: The equation for the eigenvalues of M is

$$\lambda^2 - (M_{11} + M_{22})\lambda + 1 = 0$$

Because M is real, both λ and λ^* are solutions of the quadratic. Because

$$\lambda = \frac{\mathrm{Tr}(M)}{2} \pm i\sqrt{1-(\mathrm{Tr}(M)/2)^2}$$

For $|\mathrm{Tr}\, M| < 2$, $\lambda\,\lambda^* = 1$ and so $\lambda_{1,2} = e^{\pm i\mu}$. Consequently $\cos\mu = (\mathrm{Tr}\, M)/2$. Now the following matrix is trace-free.

$$M - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\mu) = \begin{pmatrix} \frac{M_{11}-M_{22}}{2} & M_{12} \\ M_{21} & \frac{M_{22}-M_{11}}{2} \end{pmatrix}$$

Simply choose

$$\alpha = \frac{M_{11} - M_{22}}{2\sin\mu}, \qquad \beta = \frac{M_{12}}{\sin\mu}, \qquad \gamma = -\frac{M_{21}}{\sin\mu}$$

and the sign of μ to properly match the individual matrix elements with $\beta > 0$. It is easily verified that $\beta\gamma - \alpha^2 = 1$. Now

$$M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(2\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(2\mu)$$

and more generally

$$M^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(n\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(n\mu)$$