



Physics 417/517

Introduction to Particle Accelerator Physics

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

Solutions Homogeneous Eqn.



Dipole

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos((s - s_i)/\rho) & \rho \sin((s - s_i)/\rho) \\ -\sin((s - s_i)/\rho)/\rho & \cos((s - s_i)/\rho) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} 1 & s - s_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the focusing direction $k = B' / B\rho$

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}(s - s_i)) & \sin(\sqrt{k}(s - s_i))/\sqrt{k} \\ -\sqrt{k} \sin(\sqrt{k}(s - s_i)) & \cos(\sqrt{k}(s - s_i)) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the defocusing direction $k = B' / B\rho$

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{-k}(s - s_i)) & \sinh(\sqrt{-k}(s - s_i))/\sqrt{-k} \\ \sqrt{-k} \sinh(\sqrt{-k}(s - s_i)) & \cosh(\sqrt{-k}(s - s_i)) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Transfer Matrices

Dipole with bend Θ (put coordinate of final position in solution)

$$\begin{pmatrix} x(s_{\text{after}}) \\ \frac{dx}{ds}(s_{\text{after}}) \end{pmatrix} = \begin{pmatrix} \cos(\Theta) & \rho \sin(\Theta) \\ -\sin(\Theta)/\rho & \cos(\Theta) \end{pmatrix} \begin{pmatrix} x(s_{\text{before}}) \\ \frac{dx}{ds}(s_{\text{before}}) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x(s_{\text{after}}) \\ \frac{dx}{ds}(s_{\text{after}}) \end{pmatrix} = \begin{pmatrix} 1 & L_{\text{drift}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(s_{\text{before}}) \\ \frac{dx}{ds}(s_{\text{before}}) \end{pmatrix}$$

Quadrupole in the focusing direction length L

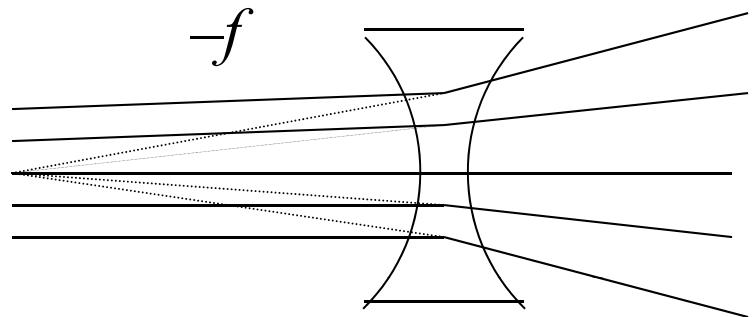
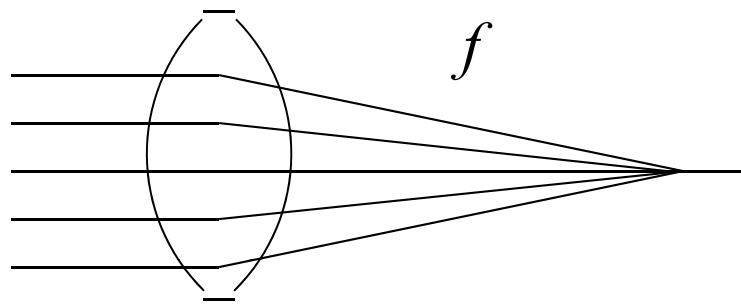
$$\begin{pmatrix} x(s_{\text{after}}) \\ \frac{dx}{ds}(s_{\text{after}}) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}L) & \sin(\sqrt{k}L)/\sqrt{k} \\ -\sqrt{k} \sin(\sqrt{k}L) & \cos(\sqrt{k}L) \end{pmatrix} \begin{pmatrix} x(s_{\text{before}}) \\ \frac{dx}{ds}(s_{\text{before}}) \end{pmatrix}$$

Quadrupole in the defocusing direction length L

$$\begin{pmatrix} x(s_{\text{after}}) \\ \frac{dx}{ds}(s_{\text{after}}) \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{-k}L) & \sinh(\sqrt{-k}L)/\sqrt{-k} \\ \sqrt{-k} \sinh(\sqrt{-k}L) & \cosh(\sqrt{-k}L) \end{pmatrix} \begin{pmatrix} x(s_{\text{before}}) \\ \frac{dx}{ds}(s_{\text{before}}) \end{pmatrix}$$

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Thin Lenses



Thin Focusing Lens (limiting case when argument goes to zero!)

$$\begin{pmatrix} x(s_{lens} + \varepsilon) \\ \frac{dx}{ds}(s_{lens} + \varepsilon) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x(s_{lens} - \varepsilon) \\ \frac{dx}{ds}(s_{lens} - \varepsilon) \end{pmatrix}$$

Thin Defocusing Lens: change sign of f

Composition Rule: Matrix Multiplication!



Element 1		Element 2
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$s_0 \qquad \qquad \qquad s_1 \qquad \qquad \qquad s_2$

$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = M_1 \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} \quad \begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M_2 \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M_2 M_1 \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

More generally

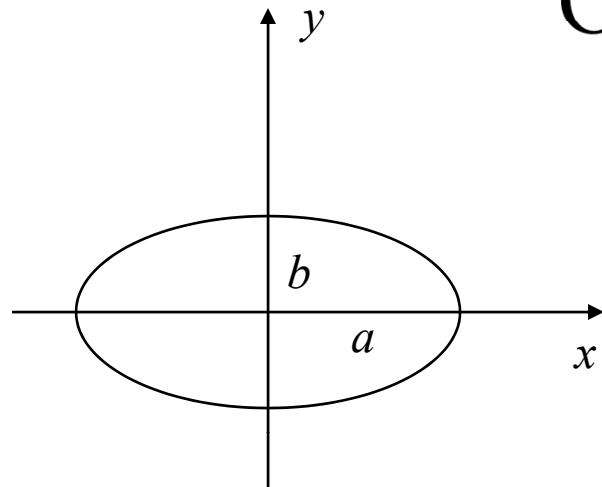
$$M_{tot} = M_N M_{N-1} \dots M_2 M_1$$

Remember: First element farthest RIGHT

Some Geometry of Ellipses

Equation for an upright ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



In beam optics, the equations for ellipses are normalized (by multiplication of the ellipse equation by ab) so that the area of the ellipse divided by π appears on the RHS of the defining equation. For a general ellipse

$$Ax^2 + 2Bxy + Cy^2 = D$$

The area is easily computed to be

$$\frac{\text{Area}}{\pi} \equiv \varepsilon = \frac{D}{\sqrt{AC - B^2}} \quad \text{Eqn. (1)}$$

So the equation is equivalently

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon$$

$$\gamma = \frac{A}{\sqrt{AC - B^2}}, \quad \alpha = \frac{B}{\sqrt{AC - B^2}}, \quad \text{and} \quad \beta = \frac{C}{\sqrt{AC - B^2}}$$

When normalized in this manner, the equation coefficients clearly satisfy

$$\beta\gamma - \alpha^2 = 1$$

Example: the defining equation for the upright ellipse may be rewritten in following suggestive way

$$\frac{b}{a}x^2 + \frac{a}{b}y^2 = ab = \varepsilon$$

$$\beta = a/b \text{ and } \gamma = b/a, \quad \text{note} \quad x_{\max} = a = \sqrt{\beta\varepsilon}, \quad y_{\max} = b = \sqrt{\gamma\varepsilon}$$