Physics 417/517 Introduction to Particle Accelerator Physics

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Solutions Homogeneous Eqn.

Dipole

$$\begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left((s-s_i)/\rho\right) & \rho\sin\left((s-s_i)/\rho\right) \\ -\sin\left((s-s_i)/\rho\right)/\rho & \cos\left((s-s_i)/\rho\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \dfrac{dx}{ds}(s_i) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(s_i) \\ \dfrac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the focusing direction $k = B'/B\rho$

$$\begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{k}\,(s-s_i)\right) & \sin\left(\sqrt{k}\,(s-s_i)\right)/\sqrt{k} \\ -\sqrt{k}\sin\left(\sqrt{k}\,(s-s_i)\right) & \cos\left(\sqrt{k}\,(s-s_i)\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \dfrac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the defocusing direction $k = B'/B\rho$

$$\begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}\,(s-s_i)\right) & \sinh\left(\sqrt{-k}\,(s-s_i)\right)/\sqrt{-k} \\ \sqrt{-k}\sinh\left(\sqrt{-k}\,(s-s_i)\right) & \cosh\left(\sqrt{-k}\,(s-s_i)\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \dfrac{dx}{ds}(s_i) \end{pmatrix}$$

Transfer Matrices

Dipole with bend Θ (put coordinate of final position in solution)

$$\begin{pmatrix} x\left(s_{after}\right) \\ \dfrac{dx}{ds}\left(s_{after}\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\Theta\right) & \rho\sin\left(\Theta\right) \\ -\sin\left(\Theta\right)/\rho & \cos\left(\Theta\right) \end{pmatrix} \begin{pmatrix} x\left(s_{before}\right) \\ \dfrac{dx}{ds}\left(s_{before}\right) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x\left(s_{after}\right) \\ \dfrac{dx}{ds}\left(s_{after}\right) \end{pmatrix} = \begin{pmatrix} 1 & L_{drift} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x\left(s_{before}\right) \\ \dfrac{dx}{ds}\left(s_{before}\right) \end{pmatrix}$$

Quadrupole in the focusing direction length L

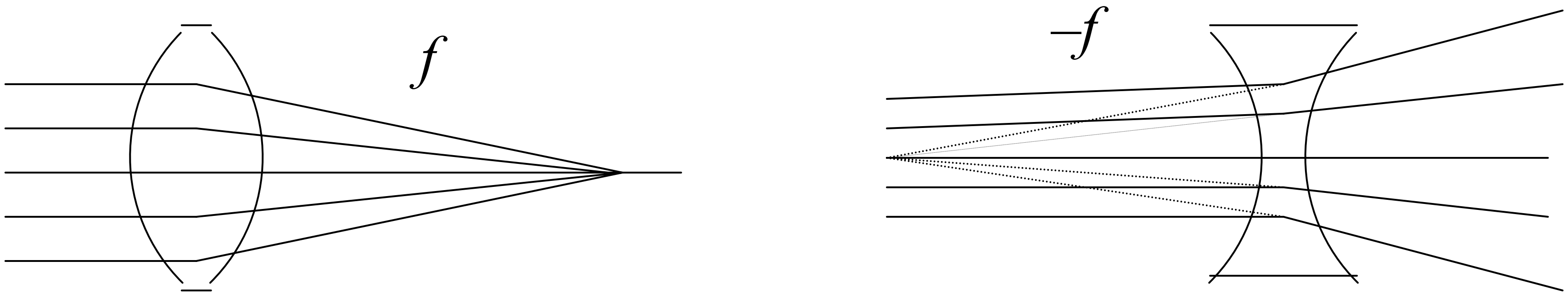
$$\begin{pmatrix} x\left(s_{after}\right) \\ \dfrac{dx}{ds}\left(s_{after}\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{k}L\right) & \sin\left(\sqrt{k}L\right)/\sqrt{k} \\ -\sqrt{k}\sin\left(\sqrt{k}L\right) & \cos\left(\sqrt{k}L\right) \end{pmatrix} \begin{pmatrix} x\left(s_{before}\right) \\ \dfrac{dx}{ds}\left(s_{before}\right) \end{pmatrix}$$

Quadrupole in the defocusing direction length L

$$\begin{pmatrix} x\left(s_{after}\right) \\ \dfrac{dx}{ds}\left(s_{after}\right) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}L\right) & \sinh\left(\sqrt{-k}L\right)/\sqrt{-k} \\ \sqrt{-k}\sinh\left(\sqrt{-k}L\right) & \cos\left(\sqrt{-k}L\right) \end{pmatrix} \begin{pmatrix} x\left(s_{before}\right) \\ \dfrac{dx}{ds}\left(s_{before}\right) \end{pmatrix}$$

Wille: pg. 71

Thin Lenses



Thin Focusing Lens (limiting case when argument goes to zero!)

$$\begin{pmatrix} x\left(s_{lens}+\varepsilon\right) \\ \dfrac{dx}{ds}\left(s_{lens}+\varepsilon\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x\left(s_{lens}-\varepsilon\right) \\ \dfrac{dx}{ds}\left(s_{lens}-\varepsilon\right) \end{pmatrix}$$

Thin Defocusing Lens: change sign of f

Composition Rule: Matrix Multiplication!

Element 1 Element 2

s_0 s_1 s_2

$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = M_1 \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} \qquad \begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M_2 \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M_2 M_1 \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

More generally

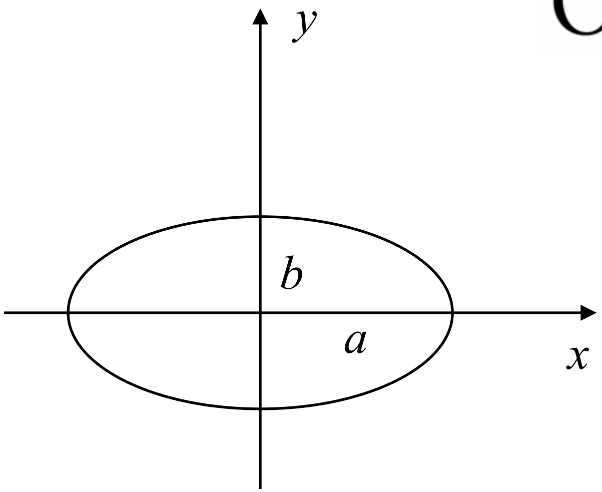
$$M_{tot} = M_N M_{N-1} \ldots M_2 M_1$$

Remember: First element farthest RIGHT

Some Geometry of Ellipses

Equation for an upright ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



In beam optics, the equations for ellipses are normalized (by multiplication of the ellipse equation by *ab*) so that the area of the ellipse divided by π appears on the RHS of the defining equation. For a general ellipse

$$Ax^2 + 2Bxy + Cy^2 = D$$

The area is easily computed to be

$$\frac{\text{Area}}{\pi} \equiv \varepsilon = \frac{D}{\sqrt{AC - B^2}} \qquad \text{Eqn. (1)}$$

So the equation is equivalently

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon$$

$$\gamma = \frac{A}{\sqrt{AC - B^2}}, \quad \alpha = \frac{B}{\sqrt{AC - B^2}}, \quad \text{and} \quad \beta = \frac{C}{\sqrt{AC - B^2}}$$

When normalized in this manner, the equation coefficients clearly satisfy

$$\beta\gamma - \alpha^2 = 1$$

Example: the defining equation for the upright ellipse may be rewritten in following suggestive way

$$\frac{b}{a}x^2 + \frac{a}{b}y^2 = ab = \varepsilon$$

$\beta = a/b$ and $\gamma = b/a$, note $x_{\max} = a = \sqrt{\beta\varepsilon}$, $y_{\max} = b = \sqrt{\gamma\varepsilon}$