Physics 417/517 Introduction to Particle Accelerator Physics

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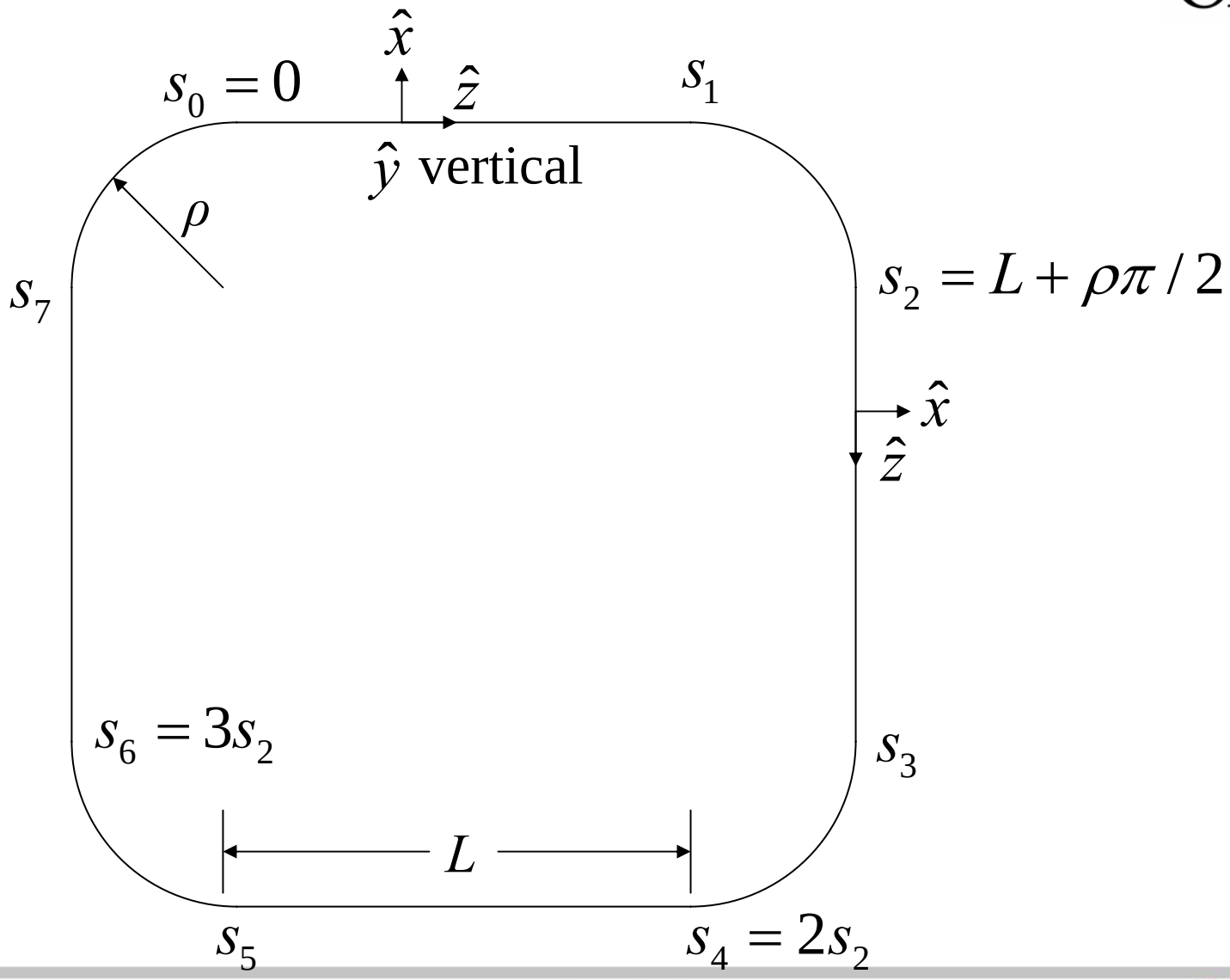
Comment on Design Trajectory

The notion of specifying curves in terms of their path length is standard in courses on the vector analysis of curves. A good discussion in a Calculus book is Thomas, *Calculus and Analytic Geometry*, 4th Edition, Articles 14.3-14.5. Most vector analysis books have a similar, and more advanced discussion under the subject of "Frenet-Serret Equations". Because all of our design trajectories involve only arcs of circles and straight lines (dipole magnets and the drift regions between them define the orbit), we can concentrate on a simplified set of equations that "only" involve the radius of curvature of the design orbit. It may be worthwhile giving a simple example.

4-Fold Symmetric Synchrotron



Its Design Trajectory

$$(0,0,s) \qquad 0 < s < L = s_1$$

$$(0,0,L) + \rho\left(\cos\left((s-s_1)/\rho\right) - 1, 0, \sin\left((s-s_1)/\rho\right)\right) \qquad s_1 < s < s_2$$

$$(-\rho, 0, L+\rho) + (s-s_2)(-1,0,0) \qquad s_2 < s < s_3$$

$$(-L-\rho, 0, L+\rho) + \rho\left(-\sin\left((s-s_3)/\rho\right), 0, \cos\left((s-s_3)/\rho\right) - 1\right) \qquad s_3 < s < s_4$$

$$(-L-2\rho, 0, L) + (s-s_4)(0,0,-1) \qquad s_4 < s < s_5$$

$$(-L-2\rho, 0, 0) + \rho\left(1-\cos\left((s-s_5)/\rho\right), 0, -\sin\left((s-s_5)/\rho\right)\right) \qquad s_5 < s < s_6$$

$$(-L-\rho, 0, -\rho) + (s-s_6)(1,0,0) \qquad s_6 < s < s_7$$

$$(-\rho, 0, -\rho) + \rho\left(\sin\left((s-s_7)/\rho\right), 0, 1-\cos\left((s-s_7)/\rho\right)\right) \qquad s_7 < s < 4s_2$$

Inhomogeneous Hill's Equations

Fundamental transverse equations of motion in particle accelerators for small deviations from design trajectory

$$\frac{d^2x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]x=\frac{1}{\rho_x(s)}\frac{\Delta p}{p}$$

$$\frac{d^2y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]y=\frac{1}{\rho_y(s)}\frac{\Delta p}{p}$$

ρ radius of curvature for bends, B' transverse field gradient for magnets that focus (positive corresponds to horizontal focusing), $\Delta p/p$ momentum deviation from design momentum. Homogeneous equation is 2nd order *linear* ordinary differential equation.

Dispersion

From theory of linear ordinary differential equations, the general solution to the inhomogeneous equation is the sum of **any** solution to the inhomogeneous equation, called the particular integral, plus two linearly independent solutions to the homogeneous equation, whose amplitudes may be adjusted to account for boundary conditions on the problem.

$$x(s)=x_p(s)+A_x x_1(s)+B_x x_2(s) \qquad y(s)=y_p(s)+A_y y_1(s)+B_y y_2(s)$$

Because the inhomogeneous terms are proportional to *Δp/p*, the particular solution can generally be written as

$$x_p(s)=D_x(s)\frac{\Delta p}{p} \qquad y_p(s)=D_y(s)\frac{\Delta p}{p}$$

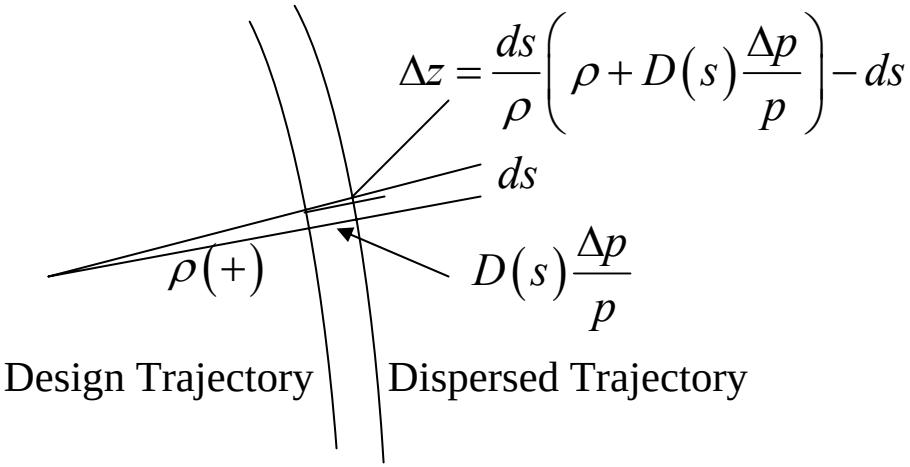
where the dispersion functions satisfy

$$\frac{d^2 D_x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]D_x=\frac{1}{\rho_x(s)} \qquad \frac{d^2 D_y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]D_y=\frac{1}{\rho_y(s)}$$

M_{56}

In addition to the transverse effects of the dispersion, there are important effects of the dispersion along the direction of motion. The primary effect is to change the time-of-arrival of the off-momentum particle compared to the on-momentum particle which traverses the design trajectory.

$$d(\Delta z) = D(s)\frac{\Delta p}{p}\frac{ds}{\rho(s)}$$



$$M_{56} = \int_{s_1}^{s_2} \left\{ \frac{D_x(s)}{\rho_x(s)} + \frac{D_y(s)}{\rho_y(s)} \right\} ds$$

Solutions Homogeneous Eqn.

Dipole

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left((s-s_i)/\rho\right) & \rho\sin\left((s-s_i)/\rho\right) \\ -\sin\left((s-s_i)/\rho\right)/\rho & \cos\left((s-s_i)/\rho\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the focusing direction $k = B'/B\rho$

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{k}\,(s-s_i)\right) & \sin\left(\sqrt{k}\,(s-s_i)\right)/\sqrt{k} \\ -\sqrt{k}\sin\left(\sqrt{k}\,(s-s_i)\right) & \cos\left(\sqrt{k}\,(s-s_i)\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Thin Focusing Lens (limiting case when argument goes to zero!)

$$\begin{pmatrix} x(s+\varepsilon) \\ \frac{dx}{ds}(s+\varepsilon) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x(s-\varepsilon) \\ \frac{dx}{ds}(s-\varepsilon) \end{pmatrix}$$

Thin Defocusing Lens: change sign of f